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COLLEGE OF ENGINEERING
DEPARTMENT OF ELECTRICAL ENGINEERING
Radiation Laboratory

Studies in Radar Cross Sections XLIX
Diffraction and Scattering by Regular Bodies III: The Prolate Spheroid

by

F.B. SLEATOR

February 1964

Project 5635
Task 563502
Contract AF 19(604)-6655

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STUDIES IN RADAR CROSS SECTIONS XLIX - DIFFRACTION AND SCATTERING BY REGULAR BODIES III: THE PROLATE SPHEROID

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F. B. Sleator

February 1964

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Prepared for

ELECTRONICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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I

INTRODUCTION

This is the third in a series of summary reports on the general subject of electromagnetic and acoustical scattering by certain bodies of simple shape. The choice of the spheroid as next in line after the sphere and cone is suggested by several considerations. The ellipsoid, of which the spheroid is a specialization, is the only remaining finite body for which 'exact' analytical solutions of boundary value problems involving the vector and scalar wave equations are at all feasible, and for the general ellipsoid these are of such complexity and tedium that few investigators have had the requisite combination of motivation and temerity to attack them. The attractions which the spheroid holds for the analyst are thus evident. Not only does it afford a generalization of all the existing work on the sphere, but the presence of an additional independent parameter offers a means of developing entirely new approximate techniques. Furthermore, the wide range of forms which can be approximated reasonably well by a spheroid includes many which are of vital interest in various fields.

The two types of spheroid, prolate and oblate, are from an exact analytical standpoint nearly identical, to the extent that, given an exact solution for one body, the corresponding solution for the other is almost trivially obtainable, at least in terms of a corresponding set of special functions. However, the prolate form

seems to predominate in the literature, partly because its limiting configuration is a thin finite rod, which is the most elementary form for an antenna. The advent of radar and the essentially prolate form of many aerodynamically efficient shapes naturally provided strong motivation for the development of this branch of the family. In the extremes of eccentricity the two forms are entirely distinct, as are the associated physical phenomena and appropriate analytical approaches, so that the oblate spheroid has a sufficiently separate entity to warrant individual consideration in a later report.

Perhaps the first problem which presents itself in the construction of a report of this nature is that of how much or what to include. In the cases treated previously the volume of literature was such that a serious problem of selection and emphasis was incurred. In the present case the volume is not so overwhelming, and this produces the initial dilemma of whether or not to try to include everything, at least in some degree of coverage. (One is reminded of the Englishman of a bygone era who purportedly remarked of the turkey that it was a most inconvenient sized bird— a little too much for one man and not quite enough for two.) The somewhat inordinate length of what follows is the result of a leaning toward the positive horn. Some sort of compromise is, however, inevitable and an element of arbitrariness is bound to enter at some poin . Thus we will limit our consideration in general to problems of diffraction or scattering where the source of energy is exterior to the scatterer (one exception is the case of a point dipole located at the tip of a spheroid, which is

immediately obtainable from a more general form). Even under this restriction the problems of accumulation and editing are non-trivial, and it is quite possible, not to say probable, that due to imperfect information or inadvertent bias some pertinent and significant work has been slighted. If such be the case, all due apologies are hereby offered and amendments invited.

Another question which must be faced at the outset is the nature of the objectives of a compendium of this type. Certainly it cannot be expected to supplant the original sources completely, and as a mere catalog of these its purpose might best be served by brevity and reduction to concise statements of conditions and results. On the other hand, in the emergence and analysis of new problems, conditions and results of the old are often of little utility, and the primary interest centers on principles and techniques. It thus appears necessary to discuss these at sufficient length to give a fairly comprehensive picture of the state of the art. At any rate, the question of the optimum degree of detail to present is an eternal and rather delicate one, and in a treatise of this length the maintenance of consistency in this respect is not easy. It is hoped that whatever its limitations, the account which follows will serve as a reasonably complete and convenient guide to existing solutions and as a catalyst in the development of new ones.

An adequate historical survey of the spheroid problem, complete to the date of its publication, is contained in Flammer's treatise on Spheroidal Wave Functions (1957). Since then several important advances have been made, notably in the

approximate or asymptotic theories for high and low frequencies. In the former range are the geometrical approach of Levy and Keller (1959) and the asymptotic solutions of Kazarinoff and Ritt (1959) for the not-too-thin body, and of Goodrich and Kazarinoff (1963) for the thin one. In the latter range is the work of Senior (1964), who has also given a comprehensive discussion of the convergence properties of the low-frequency series in general (Senior, 1961). Also of interest are the vector solutions for 'weak' scatterers given by Shatilov (1960) and Ikeda (1963), which might be considered extensions of the scalar solution of Montroll and Hart (1951). Despite these contributions, however, there is much to be done before the spheroid problem can be deemed as well understood as that of the sphere. Since the work of Schultz (1950) and the computations based on this by Siegel et al (1956), virtually no progress has been made in the solution of the vector problem in the resonance region. All existing techniques either break down completely or become prohibitively difficult or tedious in this region, and the need for a totally new approach becomes more and more apparent. Asymptotic solutions which hold for all eccentricities are still lacking, though it seems possible that the methods already developed might be extended or modified to cover the entire range. Experimental data are also strangely scarce, not only in the resonance region but at all frequencies, The few curves and points which have been assembled here are the meager fruit of an intensive literature search, and include some unpublished data as well, e.g. certain data obtained

at the Ohio State University Antenna Laboratory and at Cornell Aeronautical Laboratory.

One of the principal headaches involved in the general speroid problem is the necessity of dealing with a distinctive set of special functions, known logically enough as spheroidal functions. These have investigated quite thoroughly by several authors and are now fairly extensively tabulated, but since their properties depend on an additional parameter as compared with the spherical functions, and since there are no usable recurrence relations, the manipulation and computation of these functions is inevitably a nuisance. The first section of the next chapter deals at some length with these functions in an effort (perhaps futile) to make them appear less formidable to the uninitiated and thus facilitate the absorption of the accounts which follow. A catalog of the existing numerical tables, listing the parameter ranges and indices covered, is given in the Appendix. Another source of grief and frustration is the wide variety of notations rampant in the literature. Little can be done at this stage to standardize the notation in long-since-published works, but at least we can give a complete account and comparison of two of the most common systems and refer the reader to a fairly adequate table of these and the rest which appears in Flammer (1957). The remainder of this report is, as far as possible, consistent in the use of one of the systems detailed.

The body of the report consists of three distinct components, the first and most extensive consisting of a largely verbal discussion of the methods and principles

employed in the various solutions, the second being a tabulation of the most essential results (here again a subjective judgement is implied), and the third containing the graphical representations of these and the experimental findings. This arrangement was chosen in the hope that it might increase the overall legibility and maximize the convenience for the occasional user. The admitted disadvantages are perhaps mitigated by numerous cross-references.

The author is indebted to a number of colleagues for substantial contributions and support in the production of this report. In particular the sections containing the graphical results are almost entirely the work of Dr. R. E. Kleinman, whose constant advice and ample assistance were also instrumental in the completion of the remainder of the work. It is a pleasure also to acknowledge the faithful service of Miss K. R. Pushpamala, John Asvestas, and Soonsung Hong in the accumulation and preparation of the material, and the patient labor of Miss Mary Jane Jahnke, who typed the difficult manuscript.

II

WAVE-FUNCTION SOLUTIONS

2.1 MATHEMATICAL BACKGROUND

2.1.1 Spheroidal Geometry

The geometry of the prolate spheroidal coordinate system, which is vital to the analytical treatment of the problems we are to consider, is given in detail in many standard sources. Unfortunately there is no uniformity of notation and the many systems in use represent a major obstacle in the assimilation of material from the different sources. We will present here a fairly detailed account of two of the most widely used systems in the hope of providing at least an adequate basis for deciphering the others. The diagram in Fig. 1 shows a cross section in the Cartesian xz-plane, and the cylindrical symmetry about the z-axis completes the specification. The surfaces $\xi \equiv \cosh\mu$ = const., $\eta \equiv \cos\theta$ = const., ϕ = const. are respectively confocal prolate spheroids of major axis $2a=2F\xi=2F\cosh\mu$ and minor axis $2b=2F\sqrt{\xi^2-1}=2F\sinh\mu$, two-sheeted hyperboloids (actually one sheet corresponds to a positive η, the other to a negative), and azimuthal planes originating in the z-axis.

The two representations of spheroidal variables, (ξ, η, ϕ) and (μ, θ, ϕ), are both prevalent in the literature. While the (ξ, η, ϕ) notation is convenient in

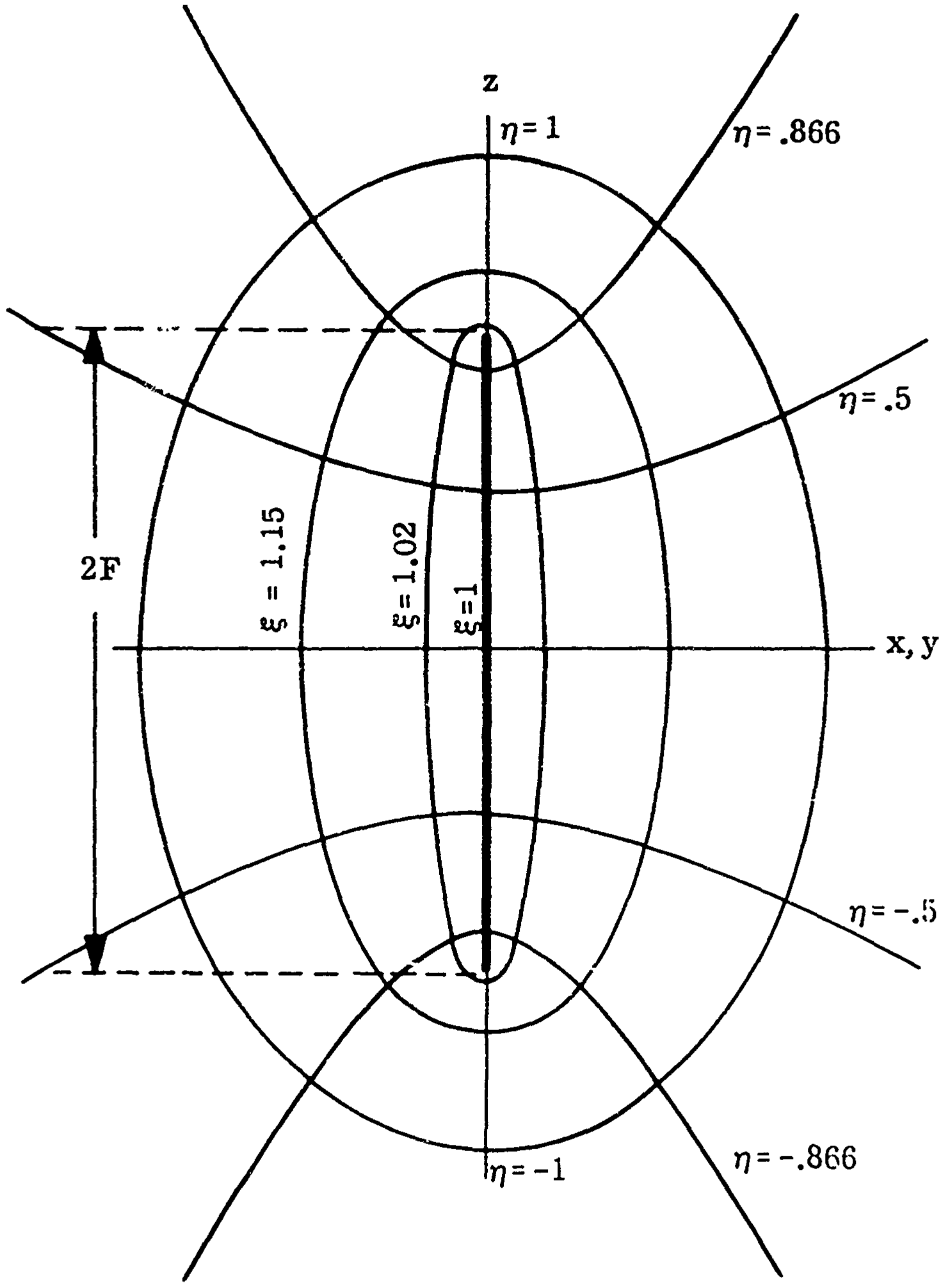

FIG. 1: THE PROLATE SPHEROIDAL COORDINATE SYSTEM.

one respect, in that a single symbol represents the arguments of the prolate spheroidal functions, the (μ, θ, ϕ) notation is convenient in another respect, giving rise to a right-handed system of coordinate vectors as opposed to the left-handed system associated with (ξ, η, ϕ). In treating scalar problems the (ξ, η, ϕ) system is, perhaps, preferable. It is certainly widely used and will be in the present work. Vector problems involving spheroids are so complicated that the use of the (μ, θ, ϕ) system may be desirable in order to avoid a left-handed system, but most of the literature employs the (ξ, η, ϕ) variables, and the present account will do likewise.

The essential relations between these coordinates and the Cartesian system may be specified by the following forms:

$$x = F\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\phi = F\sinh\mu\sin\theta\cos\phi$$

$$y = F\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\phi = F\sinh\mu\sin\theta\sin\phi$$

$$z = F\xi\eta = F\cosh\mu\cos\theta$$

where ranges are $1 \leqslant \xi \leqslant \infty$, $-1 \leqslant \eta \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$, or $0 \leqslant \mu \leqslant \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$.

Thus, in the (ξ, η, ϕ) system,

$$\frac{\partial\xi}{\partial x} = \frac{\xi}{F}\frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{(\xi^2 - \eta^2)}\cos\phi, \quad \frac{\partial\xi}{\partial y} = \frac{\xi}{F}\frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{(\xi^2 - \eta^2)}\sin\phi, \quad \frac{\partial\xi}{\partial z} = \frac{\eta}{F}\frac{(\xi^2 - 1)}{(\xi^2 - \eta^2)}$$

$$\frac{\partial\eta}{\partial x} = \frac{-\eta}{F}\frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{(\xi^2 - \eta^2)}\cos\phi, \quad \frac{\partial\eta}{\partial y} = \frac{-\eta}{F}\frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{(\xi^2 - \eta^2)}\sin\phi, \quad \frac{\partial\eta}{\partial z} = \frac{\xi(1 - \eta^2)}{F(\xi^2 - \eta^2)}$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{F\sqrt{(\xi^2 - 1)(1 - \eta^2)}}, \quad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{F\sqrt{(\xi^2 - 1)(1 - \eta^2)}}, \quad \frac{\partial\phi}{\partial z} = 0$$

and unit coordinate vectors* are related as follows:

$$\hat{i}_x = \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi \cos\phi\,\hat{i}_\xi - \sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta\cos\phi\,\hat{i}_\eta - \sin\phi\,\hat{i}_\phi$$

$$\hat{i}_y = \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi \sin\phi\,\hat{i}_\xi - \sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta\sin\phi\,\hat{i}_\eta + \cos\phi\,\hat{i}_\phi$$

$$\hat{i}_z = \sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta\,\hat{i}_\xi + \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi\,\hat{i}_\eta ,$$

and $\hat{i}_\xi = \dfrac{\sqrt{1-\eta^2}}{\sqrt{\xi^2-\eta^2}}\,\xi\cos\phi\,\hat{i}_x + \dfrac{\sqrt{1-\eta^2}}{\sqrt{\xi^2-\eta^2}}\,\xi\sin\phi\,\hat{i}_y + \dfrac{\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}}\,\eta\,\hat{i}_z$

$$\hat{i}_y = -\frac{\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}}\,\eta\cos\phi\,\hat{i}_x - \frac{\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}}\,\eta\sin\phi\,\hat{i}_y + \frac{\sqrt{1-\eta^2}}{\sqrt{\xi^2-\eta^2}}\,\xi\,\hat{i}_z$$

$$\hat{i}_\phi = -\sin\phi\,\hat{i}_x + \cos\phi\,\hat{i}_y$$

Note that $\hat{i}_\xi \wedge \hat{i}_\eta = -\hat{i}_\phi$, $\hat{i}_\eta \wedge \hat{i}_\phi = -\hat{i}_\xi$, $\hat{i}_\phi \wedge \hat{i}_\xi = -\hat{i}_\eta$,

The metric coefficients are

$$h_\xi = F\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \quad h_\eta = F\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \quad h_\phi = F\sqrt{(\xi^2-1)(1-\eta^2)}.$$

In the (μ, θ, ϕ) system

$$\frac{\partial\mu}{\partial x} = \frac{\cosh\mu\sin\theta\cos\phi}{F(\cosh^2\mu-\cos^2\theta)}, \quad \frac{\partial\mu}{\partial y} = \frac{\cosh\mu\sin\theta\sin\phi}{F(\cosh^2\mu-\cos^2\theta)}, \quad \frac{\partial\mu}{\partial z} = \frac{\sinh\mu\cos\theta}{F(\cosh^2\mu-\cos^2\theta)}$$

$$\frac{\partial\theta}{\partial x} = \frac{\sinh\mu\cos\theta\cos\phi}{F(\cosh^2\mu-\cos^2\theta)}, \quad \frac{\partial\theta}{\partial y} = \frac{\sinh\mu\cos\theta\sin\phi}{F(\cosh^2\mu-\cos^2\theta)}, \quad \frac{\partial\theta}{\partial z} = \frac{-\cosh\mu\sin\theta}{F(\cosh^2\mu-\cos^2\theta)}$$

$$\frac{\partial\phi}{\partial x} = \frac{-\sin\phi}{F\sinh\mu\sin\theta}, \quad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{F\sinh\mu\sin\theta}, \quad \frac{\partial\phi}{\partial z} = 0$$

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*As in the preceding reports of this series, a unit vector will always be denoted by a caret, all other vectors by underlined symbols. Also the vector product will be denoted by a caret, viz. $\underline{A} \wedge \underline{B}$, and the scalar product by a dot, viz. $\underline{A} \cdot \underline{B}$.

and the unit coordinate vectors are related as follows:

$$\hat{i}_x = \frac{\cosh\mu \sin\theta \cos\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\mu + \frac{\sinh\mu \cos\theta \cos\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\theta - \sin\phi\,\hat{i}_\phi$$

$$\hat{i}_y = \frac{\cosh\mu \sin\theta \sin\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\mu + \frac{\sinh\mu \cos\theta \sin\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\theta + \cos\phi\,\hat{i}_\phi$$

$$\hat{i}_z = \frac{\sinh\mu \cos\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\mu - \frac{\cosh\mu \sin\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_\theta$$

and

$$\hat{i}_\mu = \hat{i}_\xi = \frac{\cosh\mu \sin\theta \cos\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_x + \frac{\cosh\mu \sin\theta \sin\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_y + \frac{\sinh\mu \cos\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_z$$

$$\hat{i}_\theta = -\hat{i}_\eta = \frac{\sinh\mu \cos\theta \cos\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_x + \frac{\sinh\mu \cos\theta \sin\phi}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_y - \frac{\cosh\mu \sin\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}}\hat{i}_z$$

$$\hat{i}_\phi = -\sin\phi\,\hat{i}_x + \cos\phi\,\hat{i}_y$$

Note that $\hat{i}_\mu \wedge \hat{i}_\theta = \hat{i}_\phi$, $\hat{i}_\theta \wedge \hat{i}_\phi = \hat{i}_\mu$, and $\hat{i}_\phi \wedge \hat{i}_\mu = \hat{i}_\theta$.

The metric coefficients are

$$h_\mu = F\sqrt{\cosh^2\mu - \cos^2\theta}, \quad h_\theta = F\sqrt{\cosh^2\mu - \cos^2\theta}, \quad h_\phi = F\sinh\mu\sin\theta.$$

The vector operations, gradient, divergence and curl, may be expressed in terms of the metric coefficients as follows:

If ψ is a scalar function of position then

$$\nabla\psi = \frac{1}{h_\xi}\frac{\partial\psi}{\partial\xi}\hat{i}_\xi + \frac{1}{h_\eta}\frac{\partial\psi}{\partial\eta}\hat{i}_\eta + \frac{1}{h_\phi}\frac{\partial\psi}{\partial\phi}\hat{i}_\phi$$

$$= \frac{1}{h_\mu}\frac{\partial\psi}{\partial\mu}\hat{i}_\mu + \frac{1}{h_\theta}\frac{\partial\psi}{\partial\theta}\hat{i}_\theta + \frac{1}{h_\phi}\frac{\partial\psi}{\partial\phi}\hat{i}_\phi \, .$$

If $\underline{\psi}$ is a vector function of position, i.e.,

$$\underline{\psi} = \psi_\xi \hat{i}_\xi + \psi_\eta \hat{i}_\eta + \psi_\phi \hat{i}_\phi = \psi_\mu \hat{i}_\mu + \psi_\theta \hat{i}_\theta + \psi_\phi \hat{i}_\phi$$

then

$$\nabla \cdot \underline{\psi} = \frac{1}{h_\xi h_\eta h_\phi} \left[\frac{\partial}{\partial \xi} (h_\eta h_\phi \psi_\xi) + \frac{\partial}{\partial \eta} (h_\xi h_\phi \psi_\eta) + \frac{\partial}{\partial \phi} (h_\xi h_\eta \psi_\phi) \right]$$

$$= \frac{1}{h_\mu h_\theta h_\phi} \left[\frac{\partial}{\partial \mu} (h_\theta h_\phi \psi_\mu) + \frac{\partial}{\partial \theta} (h_\mu h_\phi \psi_\theta) + \frac{\partial}{\partial \phi} (h_\mu h_\theta \psi_\phi) \right]$$

and

$$\nabla \wedge \underline{\psi} = \frac{1}{h_\theta h_\phi} \left[\frac{\partial}{\partial \theta} (h_\phi \psi_\phi) - \frac{\partial}{\partial \phi} (h_\theta \psi_\theta) \right] \hat{i}_\mu + \frac{1}{h_\mu h_\phi} \left[\frac{\partial}{\partial \phi} (h_\mu \psi_\mu) - \frac{\partial}{\partial \mu} (h_\phi \psi_\phi) \right] \hat{i}_\theta$$

$$+ \frac{1}{h_\mu h_\theta} \left[\frac{\partial}{\partial \mu} (h_\theta \psi_\theta) - \frac{\partial}{\partial \theta} (h_\mu \psi_\mu) \right] \hat{i}_\phi \; .$$

Note the deliberate omission of the expression for $\nabla \wedge \underline{\psi}$ in terms of the (ξ, η, ϕ) system. This is done because, while it is true that the expressions for $\nabla \psi$ and $\nabla \cdot \underline{\psi}$ are invariant under a change of coordinate definitio the expression for $\nabla \wedge \underline{\psi}$ given above is not identical with that obtained by repl cing (μ, θ, ϕ) with (ξ, η, ϕ). That is, using $\xi = \cosh \mu$, $\eta = \cos \theta$, $\hat{i}_\xi = \hat{i}_\mu$ (which implies $\psi_\xi = \psi_\mu$), and $i_\eta = -i_\theta$ (which implies $\psi_\eta = -\psi_\theta$) together with the definitions of the metric coefficients, it is easily demonstrable that the expressions for $\nabla \psi$ and $\nabla \cdot \underline{\psi}$ in the two systems are identical. However, if we use these facts to rewrite in the (ξ, η, ϕ) system the expression for $\nabla \wedge \underline{\psi}$ given above we find that

$$\nabla \wedge \underline{\Psi} = -\frac{1}{h_\eta h_\phi}\left[\frac{\partial}{\partial \eta}(h_\phi \Psi_\phi) - \frac{\partial}{\partial \phi}(h_\eta \Psi_\eta)\right]\hat{i}_\xi - \frac{1}{h_\xi h_\phi}\left[\frac{\partial}{\partial \phi}(h_\xi \Psi_\xi) - \frac{\partial}{\partial \xi}(h_\phi \Psi_\phi)\right]\hat{i}_\eta$$

$$-\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(h_\eta \Psi_\eta) - \frac{\partial}{\partial \eta}(h_\xi \Psi_\xi)\right]\hat{i}_\phi \, .$$

Had we calculated $\nabla \wedge \underline{\Psi}$ directly in the (ξ, η, ϕ) system using the general expressions relating orthogonal coordinate systems (e.g. Magnus and Oberhettinger p. 145) we would obtain the negative of the above expression. The reversal of sense is a manifestation of the left handedness of the (ξ, η, ϕ) system. While there is nothing inherently incorrect in the consistent use of a left handed system, there is an increased probability of error when results expressed in a left handed system are compared with or transformed into right handed expressions.

2.1.2 Spheroidal Functions

The scalar Helmholtz equation $\nabla^2 \psi + k^2 \psi = 0$ written explicitly in the (ξ, η, ϕ) coordinate system becomes

$$\left[\frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial}{\partial \eta} + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)}\frac{\partial^2}{\partial \phi^2} + c^2(\xi^2 - \eta^2)\right]\psi = 0 \qquad (2.1)$$

where we have set $kF \equiv 2\pi F/\lambda \equiv c$, λ being the wavelength. The separation of this equation is accomplished in the usual way by setting

$$\psi(\xi, \eta, \phi) = U(\xi)\, V(\eta)\, W(\phi)$$

and the resulting ordinary differential equations may be written

$$\frac{d}{d\xi}\left[(\xi^2 - 1)\frac{dU}{d\xi}\right] - \left[\lambda_{mn} - c^2 \xi^2 + \frac{m^2}{\xi^2 - 1}\right] U = 0 \qquad (2.2)$$

$$\frac{d}{d\eta}\left[(1 - \eta^2)\frac{dV}{d\eta}\right] + \left[\lambda_{mn} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2}\right] V = 0 \qquad (2.3)$$

and

$$\frac{d^2 W}{d\phi^2} + m^2 W = 0 \qquad (2.4)$$

where m and λ_{mn} are the two separation constants. The functions $W(\phi)$ are thus the expected trigonometric or exponential functions, and the necessity of a single-valued representation for the field dictates that m be an integer. Specification of the λ_{mn} is more complicated and will be dealt with presently.

The theory of the spheroidal functions, which are the eigenfunctions of the second order linear ordinary differential equations (2.2), (2.3), is now fairly comprehensive, and it is not the function of this report to elucidate this in its entirety. The reader is referred to Stratton et al (1956), Meixner and Schäfke (1954), and Flammer (1957) for more detailed accounts. We will limit the present treatment to a short account of the general properties which relate these functions to the other principal families of special functions and which are needed in the applications that follow.

The hierarchy of second order differential equations to which that of the spheroidal functions most properly belongs (it is clear that the two equations (2.2), (2.3) are essentially the same, the only difference being in the range of the independent variable), is headed by Hill's equation, which is written

$$\frac{d^2 u}{dz^2} + p(z)\, u = 0 \qquad (2.5)$$

where p(z) is any real periodic function of z which can be expanded in a convergent Fourier series. If this is specialized by setting

$$p(z) = c_1 + c_2 (\mathrm{sn}\, z)^2 + c_3 k^2 (\mathrm{sn}\, z)^4 \qquad (2.6)$$

the result is a form of the Lame wave equation, which results from the separation of the Helmholtz equation in general ellipsoidal coordinates. Here sn z is a Jacobian elliptic function and if its modulus becomes unity, corresponding to deformation of the elliptic system into a prolate spheroidal one, then sn z $\longrightarrow$ tanh z, and the transformation

$$\tanh^2 z = 1 - x^2$$

reduces (2.5) to the form (2.2). One may note that in the static limit, i.e. as $k \longrightarrow 0$, equation (2.5) with p(z) as in (2.6) still retains its ellipsoidal character, and its solutions, when properly restricted, are the Lamé functions, or ellipsoidal harmonics. On the other hand, in the same limit equations (2.2), (2.3) become essentially the equation of Legendre so that the spheroidal harmonics are expressible directly in terms of Legendre functions.

One further specialization might be mentioned. The constants c_1, c_2, c_3, in (2.6) for the ellipsoidal system are such that if the modulus of sn z approaches zero then $c_3 \longrightarrow 0$, and if c_2, which depends on both this modulus and k, remains constant, the result is a form of Mathieu's equation, which governs the wave functions of the elliptic cylinder. Another form of this equation is obtainable from the

spheroidal equations (2.2), (2.3) if the separation constant m^2 is set equal to 1/4. The complete theory of Mathieu functions can thus be derived as a special case of the general theory of spheroidal functions.

The general properties of the spheroidal functions themselves are best discussed in terms of the singularities of the differential equation (2.2), which will be taken as the prototype for all the functions required. The singularities are regular ones at $\xi = \pm 1$, each with indices $\pm \frac{m}{2}$, and an irregular one at $\xi = \infty$. In any region excluding these points, the solutions of (2.2) are analytic functions of the four quantities ξ, λ_{mn}, c^2, m^2, and of order no higher than 1/2 in terms of the last three. As noted above, the necessity for single-valuedness of the functions $W(\phi)$ restricts the values of m to the integers, and for each m a fundamental system of solutions is easily established in the neighborhood of each singularity based on some prescribed initial conditions at an arbitrary regular point. In the work of Meixner and Schäfke (1954) the use of Floquet's theory in the neighborhood of ∞ leads to the establishment of a fundamental system $U_1(\xi)$, $U_2(\xi)$ such that $U_1(\xi e^{i\pi}) = e^{i\nu\pi} U_1(\xi)$, $U_2(\xi e^{i\pi}) = e^{-i(\nu+1)\pi} U_2(\xi)$, for certain values of ν, and the general solution then has the property $U(\xi e^{i\pi}) = e^{i\nu\pi} U(\xi)$. The quantity ν is called the characteristic exponent, and its permissible values are determined by the condition that

$$\sin \nu\pi = \frac{1}{2i} \left[U_1(\xi_0 e^{i\pi}) - U_2'(\xi_0 e^{i\pi}) \right]$$

where ξ_o is an arbitrary point where the initial values are specified, so that for each set of values c and m there is a denumerably infinite set of allowable values of ν, and a corresponding set of eigenvalues $\lambda_{m\nu}$. The eigenfunctions we have to deal with are thus a doubly infinite set with indices m and ν, the former indicating the order and the latter the degree. For most of the applications which follow, it is required that the functions be finite at the singularities of the differential equation $\xi = \pm 1$, and by analogy with the Legendre functions, to which the spheroidal functions must reduce in the static limit, the index ν must be an interger $\geq$ m. (An exception to this, however, will be noted in the next chapter.)

No legitimate recurrence relations (i.e., formulas which relate two or more contiguous functions in terms of coefficients which do not involve other spheroidal functions) are as yet known, and the nature of the differential equation (2.2) precludes their establishment by the usual techniques. However, the expansion of the functions in terms of other known eigenfunctions of simpler equations, e. g., Bessel, Gegenbauer, Legendre, etc., yields three-term recurrence relations for the expansion coefficients, and these form the basis of most numerical treatments of the functions. The coefficients necessarily involve the eigenvalue λ_{mn}, and convergence of the series implies the convergence of a certain continued fraction, or equivalently the vanishing of an infinite determinant, which furnishes a transcendental equation that may be used to determine λ_{mn} explicitly. A more detailed account of the procedure follows presently.

It is apparent that the functions of interest must be either even or odd about the origin. This follows from the fact that since the indices at the singularities $\xi = \pm 1$ are $\pm \frac{m}{2}$, no two solutions which are finite at both these points can be linearly independent, and the continuity of the function and its derivative at the origin then requires that $U(-\xi) = \pm U(\xi)$.

The convergent representation of the solutions of (2.2) over the entire infinite range of the independent variable requires at least two distinct expansions. For the range $|\xi| \leq 1$, for which equation (2.3) is the appropriate form, an expansion in the Legendre functions $P_n^m(\eta)$ is indicated, and since the range is that of the angular variable η, the corresponding solutions are called angle functions and will be denoted hereafter by the symbol $S_{mn}(c, \eta)$. The angle functions are actually of two kinds, those which are finite at $\eta = \pm 1$ and those which become infinite there; the latter are of no utility in the physical problems to be considered, and we limit our discussion to the former, remarking only that there are analogous expressions for the latter involving the Legendre functions $Q_n^m(\eta)$.

We write then

$$S_{mn}(c, \eta) = \sum_{r=0,1}^{\infty}{}' \, d_r^{mn}(c) \, P_{m+r}^m(\eta) \tag{2.7}$$

where the prime indicates, as always hereafter, that the summation index runs over the even or odd integers according as n - m is even or odd. Substitution of this expansion in the differential equation (2.3), followed by application of the

differential equation and recurrence relations for the Legendre functions, gives the following recurrence formula for the expansion coefficients $d_r^{mn}(c)$:

$$\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)}\, d_{r+2}^{mn}(c) + \Big[(m+r)(m+r+1) - \lambda_{mn}(c) + \tag{2.8}$$

$$+ \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r)(2m+2r+3)}\, c^2\Big]\, d_r^{mn}(c) + \frac{r(r-1)\,c^2}{(2m+2r-3)(2m+2r-1)}\, d_{r-2}^{mn}(c) = 0, \; (r \geq 0).$$

There are two non-trivial solutions, i.e. sets of coefficients which satisfy this family of equations, only one of which, however, yields a convergent series in (2.7), and in this one the ratio d_r^{mn}/d_{r-2}^{mn} approaches zero as $-c^2/4r^2$. Rewriting (2.8) in terms of this ratio, iterating for the requisite range of values of r, and applying the above condition as $r \to \infty$ and the fact that $d_r^{mn} = 0$ for $r < 0$,* yields finally the transcendental equation for $\lambda_{mn}(c)$ mentioned earlier. Once this quantity is determined, the expansion coefficients may be computed in terms of an arbitrary initial value and the resulting series (2.7) will converge absolutely for all finite values of η. In practice the solution of the transcendental equation for $\lambda_{mn}(c)$ is usually accomplished by an iterative procedure using a first approximation given by a power series representation in the variable c^2, the first few coefficients of which are given in the standard literature, e.g. Flammer (1957). The arbitrary value

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*The analogous treatment of the functions of the second kind requires the definition of non-vanishing coefficients d_{-r}^{mn} with $0 < r \leq 2m$ (see Flammer, 1957). No ambiguity results, however.

mentioned above determines the normalization of the functions, and this will be fixed in the present work, as in that of Flammer and others, by the stipulation that $S_{mn}(c,0) = P_n^m(0)$ for n - m even and $\frac{d}{d\eta} S_{mn}(c,0) = \frac{d}{d\eta} P_n^m(0)$ for n - m odd, with the result that

$$\sum_{r=0}^{\infty}{}' \frac{(-1)^{\frac{r}{2}}(r+2m)!\, d_r^{mn}}{2^r\left(\frac{r}{2}\right)!\left(\frac{r+2m}{2}\right)!} = \frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^{n-m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!} \quad \text{for } (n-m) \text{ even} \tag{2.9}$$

$$\sum_{r=1}^{\infty}{}' \frac{(-1)^{\frac{r-1}{2}}(r+2m+1)!\, d_r^{mn}}{2^r\left(\frac{r-1}{2}\right)!\left(\frac{r+2m+1}{2}\right)!} = \frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)}{2^{n-m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!} \quad \text{for } n-m \text{ odd.} \tag{2.10}$$

The general Sturm-Liouville theory provides that the functions $S_{mn}(c,\eta)$ for fixed m are orthogonal over the interval $-1 \leq \eta \leq 1$ and the normalization factor N_{mn} is easily found to be

$$N_{mn} \equiv \int_{-1}^{1} \left[S_{mn}(c,\eta)\right]^2 d\eta = 2 \sum_{r=0,1}^{\infty}{}' \frac{(r+2m)!\,(d_r^{mn})^2}{(2r+2m+1)r!} \tag{2.11}$$

The index m is in general positive, but if the exponential form of the ϕ - dependence is used, it may assume negative values also, and the corresponding angle functions are related to those with m > 0 by the form

$$S_{-mn}(c,\eta) = (-1)^m \frac{(n-m)!}{(n+m)!} S_{mn}(c,\eta) \tag{2.12}$$

Useful representations for the range $\xi > 1$, i.e., for solutions of equation (2.2), can be obtained now from the form (2.7) by utilizing the fact that any solution of the Helmholtz equation forms a suitable kernel for an integral representation of one of the separated solutions in terms of another (cf. Morse and Feshbach, 1953, p. 636). If we choose as the kernel the function

$$K(\xi, \eta) = e^{ic\xi\eta} \left[(\xi^2 - 1)(1 - \eta^2)\right]^{\frac{m}{2}} ,$$

multiply this by $S_{mn}(c, \eta)$ and integrate between limits such that the bilinear concomitant vanishes at both, the result is a solution of equation (2.2) with independent variable ξ, which is called a radial function and will be denoted hereafter by $R_{mn}(c, \xi)$. Examination of the bilinear concomitant shows that there are three possible sets of limits, namely -1 and 1, $i\infty$ and 1, -1 and $i\infty$. Substitution of the expansion (2.7) for $S_{mn}(c, \eta)$ followed by use of the differential definition of the associated Legendre function $P_n^m(\eta)$ and an r-fold integration by parts leaves us with integrals of the form

$$\int_a^b e^{ic\xi\eta}(1-\eta)^{m+r} d\eta$$

and when a and b are replaced by the above three sets of limits, these integrals can be evaluated in terms of spherical Bessel functions. We are thus led to the expansions

$$R_{mn}^{(\ell)}(c,\xi) = \frac{\rho_{mn}(\xi^2-1)^{m/2}}{(c\xi)^m} \sum_{r=0,1}^{\infty}{}' d_r^{mn}\, i^r \frac{(2m+r)!}{r!} z_{m+r}^{(\ell)}(c\xi) \tag{2.13}$$

where $z_n^{(\ell)}$ is one of the four spherical Bessel functions j_n, n_n, $h_n^{(1)} \equiv j_n + in_n$, $h_n^{(2)} \equiv j_n - in_n$, accordingly as $\ell = 1, 2, 3, 4$. The normalization factor ρ_{mn} is arbitrary, and following Flammer (1957), we specify it as

$$\rho_{mn} = i^{m-n} c^m \left[\sum_{r=0,1}^{\infty}{}' d_r^{mn} \frac{(2m+r)!}{r!} \right]^{-1} \tag{2.14}$$

which gives the radial functions the same asymptotic behavior as the corresponding spherical functions for large argument, i.e.

$$R_{mn}^{(1)}(c,\xi) \xrightarrow[\xi \to \infty]{} \frac{1}{c\xi} \cos\left[c\xi - \frac{1}{2}(n+1\pi)\right], \text{ etc}^{*} \tag{2.15}$$

With this normalization the Wronskian of the first two types is easily found to be

$$\Delta(1,2) = \frac{1}{c(\xi^2-1)} . \tag{2.16}$$

If the region of definition of either the angle function S_{mn} or the radial function $R_{mn}^{(\ell)}$ is extended, with proper adjustment of the phase of any radical involved, it becomes apparent that for some ℓ, the two functions must be linearly dependent. With the definitions established above, we can thus write

$$S_{mn}(c,z) = k_{mn}^{(1)}(c)\, R_{mn}^{(1)}(c,z) \tag{2.17}$$

*The statement prevalent in the literature that this limit obtains as $c\xi \rightarrow \infty$ is not correct. For $\xi > 1$ and $c \rightarrow \infty$ the behavior is otherwise (see Silbiger, 1961).

with a similar expression relating the second type of angle function mentioned earlier to $R_{mn}^{(2)}(c, z)$. The constants of proportionality or joining factors $k_{mn}^{(1)}(c)$ may be found in terms of the coefficients d_r^{mn} by comparing the functions or their derivatives at $z = 0$. For the functions of the second type, a Laurent series may be developed in the region $1 < z < \infty$ and the coefficients of like powers of the variable equated. The forms thus obtained are given in the standard literature and will be deferred here until required.

Many other representations, characterizations, approximate forms, etc., are known for the spheroidal functions, but it is doubtful whether our present purposes would be served by dwelling on them at this point. The reader is referred to the sources mentioned above and to others cited in later sections. We close this section with a few general remarks which may contribute to the overall perspective.

To date it has not proved possible to find any elementary integral expressions for the spheroidal functions, i.e. expressions of the individual functions in terms of definite integrals involving only elementary, or even only simpler functions. They can however be characterized as solutions of linear homogeneous integral equations of the form

$$f(z) = \lambda \int K(z, z')f(z')dz' \tag{2.18}$$

where the kernel $K(z, z')$ involves only more elementary functions, as illustrated above in the derivation of the expansion for the radial functions. Other permissible

kernels, most of them involving Bessel functions, are given in the references already cited. Use of kernels involving spheroidal functions has yielded a number of definite integrals of products of these functions, see for example Chako (1955).

Other useful representations which are developed at length in the literature include power series expansions about the origin and about the singularities ± 1. In the appropriate ranges of the independent variables, these are more convenient for computation than the expansions given above. For the regions of low frequency or small eccentricity, certain expansions in powers of the parameter c have been derived, though the range of convergence of these is in general quite limited. This question has been examined by Senior (1961). Asymptotic forms valid for large c are also available and can be used to advantage in the high frequency ranges. These are in general based on the parabolic cylinder or Whittaker functions, whose equation the spheroidal equations resemble in the limit of large c. However, there are still regions in the frequency-eccentricity space, which cannot be treated conveniently by any of the representations known at present. These will be discussed in a later section.

The lack of legitimate recurrence relations for the spheroidal functions was mentioned earlier. A number of so-called recurrence relations of the Whittaker type are indeed known, but the coefficients which multiply the neighboring functions contain integrals involving other spheroidal functions, which are in general intractable, and the formulas have so far been of little utility.

The theory outlined in this section has been based exclusively on the solution of the scalar Helmholtz equation. The treatment of vector problems of course requires numerous additional concepts and derivations in most of which, however, the scalar solutions are intimately involved. The vector solution will form a separate section of this chapter.

2.2 SCALAR SOLUTIONS

2.2.1 Scalar Green's Functions

We turn now to the solution of a certain class of problems which might be interpreted physically as the scattering or radiation of time-harmonic sound waves in a homogeneous, isotropic, non-dissipative medium, by a closed prolate spheroidal surface with various types of boundary condition. The technique used is the straightforward (if sometimes tedious) method of formally expanding the requisite field quantities in series of the appropriate eigenfunctions (in this case the spheroidal functions discussed in the previous section), and determining the expansion coefficients by application of the boundary conditions at the surface and (if necessary) at infinity. The resulting solutions will be referred to as 'exact', primarily to distinguish them from the various approximate results to be taken up later on. It is understood, of course, that since these 'exact' solutions contain infinite series, their exactitude depends on the convergence properties of these series and in any practical sense, i.e. in the absence of a virtually infinite computational capacity, the achievable accuracy, particularly in the optics region, may be far less than that given by a suitable approximate technique.

Because of the orthogonality of the angular functions of both the η and ϕ variables, the procedures required here are no more complicated than those used in the case of the sphere, and the forms of the resulting solutions are directly analogous. The completeness of the angle functions, which was not specified in the previous section, follows from that of the spherical harmonics by a fairly simple argument (cf. Siegel et al 1953).

For problems of scattering or diffraction in which the energy is supplied by a source exterior to the spheroid the discussion will be limited to the case of an elementary point source at an arbitrary location. This includes the plane wave with arbitrary direction of propagation as a limiting form. The more complicated case of a dipole source will be considered later in this section when certain electro-magnetic problems which are essentially of a scalar nature are taken up. The boundary condition for the scalar problem is in general the vanishing of a linear combination of the field quantity, which is usually the sound pressure or velocity potential, and its normal derivative on the scattering surface. The particular cases of the Dirichlet condition (where the function itself vanishes and the surface is termed 'soft') and the Neumann condition (where the normal derivative vanishes and the surface is called 'hard') are both obtainable by specializing the coefficients in this linear combination.

The solution for an elementary source with any of these boundary conditions is properly termed a Green's function, and its derivation follows the standard

procedures given in any text on mathematical physics. In terms of the spheroidal coordinates specified in the previous section, the field strength at the point $\underline{r}(\xi,\eta,\phi)$ due to the unit source at the point $\underline{r}_1(\xi_1,\eta_1,\phi_1)$, i.e. the free-space Green's function,* is

$$\psi(r) = G_o(\underline{r}, \underline{r}_1) = \frac{-e^{ik|\underline{r}-\underline{r}_1|}}{4\pi|\underline{r}-\underline{r}_1|} . \tag{2.19}$$

This is then the solution of the inhomogeneous wave equation

$$\nabla^2 G + k^2 G = \delta(\underline{r} - \underline{r}_1) \tag{2.20}$$

where the right hand side is the Dirac delta function, which vanishes except at $\underline{r} = \underline{r}_1$ and whose volume integral over the entire space is unity. At large distances from the source, ψ represents a spherically diverging wave, in accordance with the well-known Sommerfeld radiation condition, which is one of the boundary conditions necessary to determine the solution of any such problem uniquely. Since the quantity $G_o(\underline{r}, \underline{r}_1)$ is symmetrical in $\underline{r}$ and $\underline{r}_1$ and satisfies the homogeneous wave equation at all points except $\underline{r} = \underline{r}_1$, the standard theory for such equations permits us to write almost immediately the formal expansion (cf. for example, Morse and Feshbach, 1953)

$$G_o(\underline{r}, \underline{r}_1) = G_o(\xi, \eta, \phi; \xi_1, \eta_1, \phi_1)$$

$$(2.21)$$

$$= \sum_{m,n} A_{mn} S_{mn}(c,\eta) S_{mn}(c,\eta_1) \cos m(\phi - \phi_1) R^{(1)}_{mn}(c,\xi_<) R^{(3)}_{mn}(c,\xi_>) ,$$

*The harmonic time dependence, $e^{-i\omega t}$, is assumed, and this factor is deleted from all field quantities.

where

$$\xi_< = \begin{matrix} \xi \\ \xi_1 \end{matrix} \quad \text{for} \quad \begin{matrix} \xi < \xi_1 \\ \xi_1 < \xi \end{matrix}$$

and conversely for $\xi_>$.

The occurrence of $R^{(3)}$ rather than $R^{(4)}$ is a direct consequence of the form of radiation condition dictated by the choice of time dependence, $\lim_{r \to \infty} r \left[\frac{\partial G}{\partial r} - ikG \right] = 0$.

By integrating both sides of equation (2.21) over a vanishingly small interval in ξ about the point ξ_1 and making use of the orthogonality of the functions of η and ϕ, the coefficients A_{mn} are found to be

$$A_{mn} = \frac{ik\epsilon_m}{2\pi N_{mn}}$$

where ϵ_m is the Neumann number, defined as

$$\epsilon_m = 1 \text{ for } m = 0$$

$$\epsilon_m = 2 \text{ for } m = 1, 2, 3 \text{ . . . .}$$

and N_{mn} is the normalization integral given in equation (2.11).

The analogous form for the total field exterior to a spheroidal boundary to which the source is also exteri..r can be obtained from (2.21) by simply adding a symmetric function of the source and observation points such that the total field satisfies the boundary condition specified on the spheroid. We consider here a linear homogeneous mixed boundary condition of the form

$$\left[\alpha\psi + \beta\,\frac{\partial\psi}{\partial n}\right]_{\xi=\xi_o} = 0 \tag{2.22}$$

where ψ is the total field due to the point source in the presence of the spheroid $\xi=\xi_o$ and $\frac{\partial}{\partial n}$ is the normal derivative, $\frac{1}{h\xi}\,\frac{\partial}{\partial\xi}$. The Green's function satisfying the condition (2.22) is then written

$$G(\underline{r},\underline{r}_1) = \frac{ik}{2\pi}\sum_{m,n}\frac{\epsilon_m}{N_{mn}}\,S_{mn}(c,\eta)S_{mn}(c,\eta_1)\cos m(\phi-\phi_1)$$
$$\cdot R^{(3)}_{mn}(c,\xi_>)\left[R^{(1)}_{mn}(c,\xi_<) - C_{mn}R^{(3)}(c,\xi_<)\right]$$

and constant C_{mn} is found by subjecting G (or more specifically, the quantity in the brackets in G) to the conditions in (2.22). Thus, we obtain

$$G(\underline{r},\underline{r}_1) = \frac{ik}{2\pi}\sum_{m,n}\frac{\epsilon_m}{N_{mn}}\;S_{mn}(c,\eta)\,S_{mn}(c,\eta_1)\cos m(\phi-\phi_1)\,\cdot$$
$$\cdot R^{(3)}_{mn}(c,\xi_>)\left[R^{(1)}_{mn}(c,\xi_<) - \frac{\alpha R^{(1)}_{mn}(c,\xi_o)+\beta\frac{\partial}{\partial n}R^{(1)}_{mn}(c,\xi_o)}{\alpha R^{(3)}_{mn}(c,\xi_o)+\beta\frac{\partial}{\partial n}R^{(3)}_{mn}(c,\xi_o)}\,R^{(3)}_{mn}(c,\xi_<)\right]. \tag{2.23}$$

The solutions for Dirichlet and Neumann boundary conditions follow immediately on setting β and α respectively equal to zero.

The solution for plane wave incidence is also obtainable from (2.23) by letting the source point recede to infinity in some arbitrary direction specified

by the spherical coordinates θ_1, ϕ_1. The resulting behavior of the affected quantities in (2.23) is as follows:

$\eta_1 \longrightarrow \cos\theta_1$, where θ_1 is the polar angle,

$$\xi_1 \longrightarrow \infty, \quad R^{(3)}_{mn}(c,\xi_1) \longrightarrow \frac{e^{i(c\,\xi_1 - (n+1)\frac{\pi}{2})}}{c\,\xi_1} = \frac{e^{ikr_1}}{kr_1}\cdot(-i)^{n+1}$$

$c\xi_1 \longrightarrow kr_1$ where r_1 is the distance of the source from the origin.

The expression for the total field must be renormalized, i.e. multiplied by the factor $r_1 e^{-ikr_1}$ in accordance with the usual plane wave representation, and the final result, which is no longer properly termed a Green's function, but which in consideration of its similarity to the previously derived expressions we might denote by the symbol G_∞, is the form

$$G_\infty(\xi,\eta,\phi;\theta_1,\phi_1) = 2\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(-i)^n\frac{\epsilon_m}{N_{mn}}S_{mn}(c,\eta)S_{mn}(c,\cos\theta_1)\cos m(\phi-\phi_1)$$

$$\cdot\left[R^{(1)}_{mn}(c,\xi) - \frac{\alpha R^{(1)}_{mn}(c,\xi_o)+\beta\frac{\partial}{\partial n}R^{(1)}_{mn}(c,\xi_o)}{\alpha R^{(3)}_{mn}(c,\xi_o)+\beta\frac{\partial}{\partial n}R^{(3)}_{mn}(c,\xi_o)}R^{(3)}_{mn}(c,\xi)\right]. \qquad (2.24)$$

One further specialization is worth noting here. If the source is restricted to the axis of symmetry of the scatterer, then $\eta_1 = 1$ and ϕ_1 disappears. From the representation (2.7) of the angle functions and the well known properties of the Legendre functions, it follows that $S_{mn}(c,1) = 0$ for $m > 0$. One of the

summations in (2.21) thus disappears along with the ϕ-dependence and the resulting form is

$$G_o(\xi, \eta; \xi_1, 1) = \frac{-ik}{2\pi} \sum_{n=o}^{\infty} \frac{1}{N_{on}} S_{on}(c, \eta) S_{on}(c, 1) R^{(1)}_{on}(c, \xi_<) R^{(3)}_{on}(c, \xi_>) , \quad (2.25)$$

and the same modifications obtain in the expressions for G and G_∞. If the observation point is in the far zone, the asymptotic forms for $R^{(1)}_{mn}(c, \xi)$ or $R^{(3)}_{mn}(c, \xi)$ may be used in (2.25) and further simplication will result. The specific forms are presented in section 4.1.

The standard problems of radiation from a spheroid can be handled in similar fashion. Here the incident field is absent and the boundary condition is inhomogeneous. A mixed linear boundary condition similar to (2.22) is generally enough to include most problems of practical interest and we outline the procedure briefly, deferring specific cases for later treatment

Suppose that

$$\left[\alpha\psi + \beta \frac{\partial\psi}{\partial n}\right]_{\xi = \xi_o} = F(\eta, \phi) \quad (2.26)$$

where F is sufficiently well-behaved so that it can be expanded in a double series of the surface wave functions $S_{mn}(c, \eta) \cos m \phi$. Then we write

$$F(\eta, \phi) = \sum_{m, n} A_{mn} S_{mn}(c, \eta) \cos m \phi \quad (2.27)$$

and from the orthogonality of the angle functions, the coefficients A_{mn} are given by

$$A_{mn}(c) = \frac{1}{N_{mn}(1+\delta_{om})\pi} \int_{-1}^{1} F(\eta, \phi) S_{mn}(c, \eta) \cos m\phi \, d\eta d\phi \tag{2.28}$$

The radiated field $\psi(\xi, \eta, \phi)$ is expanded in the same manner as before,

$$\psi(\xi, \eta, \phi) = \sum_{m,n} B_{mn} R^{(3)}_{mn}(c, \xi) S_{mn}(c, \eta) \cos m\phi \tag{2.29}$$

and subjected to the boundary condition (2.26), and since the angle functions are linearly independent we can immediately equate the coefficients of the functions $S_{mn}(c, \eta) \cos m\phi$ to give

$$B_{mn} = A_{mn} \left[\alpha R^{(3)}_{mn}(c, \xi_o) + \beta \frac{\partial}{\partial n} R^{(3)}_{mn}(c, \xi_o) \right]^{-1} \tag{2.30}$$

The obvious specializations of this result may be carried out as in the previous forms.

2.2.2 Pseudo-Scalar Problems

The formulas developed in the preceding paragraphs are sufficient for most problems involving the scattering or radiation of a time-harmonic scalar field by a spheroid of fairly arbitrary surface characteristics or behavior. We now wish to show how they can also be used to solve a limited class of vector problems in which the vector field is essentially characterized by a single scalar quantity. As will be seen, the scalar problems involved are of interest only in connection

with the vector problem from which they are derived. Hence, the designation "pseudo-scalar" will be used to distinguish them from scalar physical problems.

The general vector problem is that of finding the electric and magnetic field vectors, $\underline{E}$ and $\underline{H}$, external to a prolate spheroid in the presence of any of various incident or primary fields. Our attention will be largely restricted to bodies which are either perfect dielectrics or perfect conductors, imbedded in homogeneous, isotropic, perfectly dielectric media of permeability μ and permittivity ϵ. In MKS units the homogeneous Maxwell equations which govern the behavior of the field quantities at all ordinary points in space, are written

$$\begin{aligned} &\nabla \wedge \underline{E} - i\omega\mu \underline{H} = 0 \\ &\nabla \wedge \underline{H} + i\omega\epsilon \underline{E} = 0 \\ &\nabla \cdot \underline{E} = \nabla \cdot \underline{H} = 0. \end{aligned} \tag{2.31}$$

The expression of Maxwell's equations or the concomitant vector wave equations in the spheroidal coordinate system results in a set of three partial differential equations in the components of either the electric or the magnetic field vector, each of which contains all three components (cf. Page, 1944), and the simultaneous solution of these is in general impractical. The solution of the general vector problem must accordingly be attacked by means of a different technique, which will be described in the next section. In certain special cases, however, notably those in which the entire system is symmetric about the axis of rotation of

the spheroid, the equations degenerate, and the entire field representation can be obtained in terms of a single scalar quantity which satisfies the scalar wave equation with the ϕ-dependence removed.+ Page also points out that, as in the spherical coordinate system, the component equations separate in cases where both the electric and magnetic fields are normal to the radius vector at every point, i.e. the propagation vector is radial at every point, which is the so-called TEM mode. It should be noted, however, that this restriction is so stringent as to exclude practically all radiation or scattering problems of real interest.

The separability in the axially symmetric spheroid problem was first exploited by Abraham (1898) to find the characteristic frequencies and decay rates in a dielectric medium surrounding a conducting spheroid, and has since been used by various authors for related problems, as outlined below.

From an analytical standpoint, there are two possible types of axially symmetric field, one in which the E vector is in the meridian plane at every point and the H vector is normal to this plane, and the other in which the roles are interchanged. We limit our discussion to the former case. That is, the magnetic field is assumed to be given by

$$\underline{H} = \hat{\phi} H_\phi = -\hat{i}_x \sin\phi \, H_\phi + \hat{i}_y \cos\phi \, H_\phi \tag{2.32}$$

+Actually this holds not only in the spheroidal case but in any reasonably well-defined orthogonal coordinate system.

where H_ϕ is independent of ϕ. Since Maxwell's equations (2.31) imply that the rectangular field components satisfy the scalar wave equation, i.e.,

$$(\nabla^2 + k^2) \begin{matrix}\sin\phi \\ \cos\phi\end{matrix} H_\phi = 0,$$

and since this equation is separable (see Sec. 2.1.2) it follows that H_ϕ is proportional to the product $R_{1n}(c,\xi)S_{1n}(c,\eta)$. Note the appearance here of the spheroidal functions of order one, in contrast to the scalar case where an axially symmetric field entails only the zero order.

The mechanism is thus established for the determination of the characteristic electrical oscillations of the conducting spheroid in a dielectric medium and the solution of related boundary value problems involving axially symmetric excitation. For the former, we can apply the appropriate boundary condition to each harmonic, i.e. for each value of n, individually. On the surface of a perfect conductor the tangential electric field must vanish identically, and in our case this is simply the condition $E_\eta = 0$ at $\xi = \xi_o$. From the second equation of (2.31) and the representation of H_ϕ specified above, this is equivalent to

$$\frac{\partial}{\partial \xi}\left[\sqrt{\xi^2-1}\; R_{1n}(c,\xi)\right] = 0 \text{ at } \xi = \xi_o \qquad (2.33)$$

which is an implicit equation in the quantity $c \equiv kF$. The roots of this equation will, in general, be complex. The proof of this will not be given here, but an analogy can be drawn with the spherical Bessel functions and the essential argument is as follows. If the radiation condition is to be satisfied for large ξ, then the radial

function must be of the third type, $R^{(3)} = R^{(1)} + iR^{(2)}$. $R^{(1)}$ and $R^{(2)}$ are both real, and as in the case of Bessel functions, neither they nor their respective derivatives have any common zeros. If the root of (2.33) with least absolute value is called c_n, then the characteristic wavelength of the n^{th} harmonic is

$$\lambda_n = 2\pi F / \mathrm{Re}\, c_n$$

and the logarithmic decrement, which determines the time rate of decay of the field, is

$$\delta_n = -2\pi \,\mathrm{Im}\, c_n / \mathrm{Re}\, c_n .$$

This is essentially the procedure used by Abraham (1898), Page and Adams (1938), Page (1944), and Ryder (1942) to investigate the resonance phenomena associated with thin conducting spheroids in general, and in particular with the limiting case of a thin finite wire ($\xi_o \rightarrow 1$). The roots of (2.33) are found by expanding the radial functions and their derivatives in power series and then using a successive-approximation scheme to solve for c_n. The same techniques can be used for the case where there is an axially symmetric applied field. This field is assumed to consist of a known component of each harmonic, and the boundary condition is applied to each harmonic of the total electric field, i.e. the sum of the applied and radiated fields. In this way the above authors gained considerable quantitative and qualitative information about the resonant frequencies and decay factors of thin spheroids, as well as the antenna currents and impedances of

these bodies when stimulated by time harmonic uniform fields or plane waves of low frequency with electric vectors in the axial direction. A thorough discussion of their results in beyond the scope of this report, but certain ones of particular interest will be mentioned in a later section.

The general axially symmetric vector scattering problem can be solved in much the same way as the acoustical problem. Given the completeness of the spheroidal functions in the established ranges of the variables, the applied (or incident) and the radiated (or scattered) fields can both be expressed in terms of scalar quantities satisfying the Helmholtz equation, and these can be expanded in terms of the appropriate spheroidal functions. The known and unknown coefficients can be related as before by using the boundary conditions at the surface and the orthogonality properties of the angular functions. In this way the problem of an axial dipole located at the tip of a conducting spheroid has been solved for several eccentricities and frequencies by Hatcher and Leitner (1954), and that of the same source located at an arbitrary point on the axis for a somewhat larger range of frequencies and eccentricities by Belkina (1957). The procedure is as follows.

If a point electric dipole is oriented parallel to the axis of symmetry of the chosen coordinate system and located in this axis, then the associated scattering problems involving symmetrical bodies can be solved in terms of the single magnetic field component H_ϕ. The first step in the spheroid problem is to expand this component of the dipole field alone in series of spheroidal functions. If the dipole

moment is $p\,\hat{i}_z$ and its location is at the point $(\xi_1, 1)$ the field component at the point (ξ, η) at a distance R from $(\xi_1, 1)$, is (see, for example, Stratton 1941)

$$H_\phi = -\frac{\omega k p}{4\pi} \frac{e^{ikR}}{R} \left(1 - \frac{1}{ikR}\right) \sin\Theta \tag{2.34}$$

where

$$R = F\sqrt{\xi^2 + \eta^2 - 1 + \xi_1^2 - 2\,\xi\xi_1\eta}$$

and Θ is the angle between the vector $\underline{R}$ and the dipole axis, i.e.

$$\sin\Theta = \frac{F\sqrt{(\xi^2-1)(1-\eta^2)}}{R}$$

As in the previous cases of electromagnetic oscillations, the appropriate spheroidal functions are those of order 1, and in terms of the undetermined coefficients $\alpha_n(\xi_1)$, we write, for $\xi > \xi_1$,

$$\frac{e^{ikR}}{R}\left(1 - \frac{1}{ikR}\right) \sin\Theta = \sum_{n=0}^{\infty} \alpha_n(\xi_1) R_{1n}^{(3)}(c, \xi)\, S_{1n}(c, \eta) \tag{2.35}$$

The determination of the $\alpha_n(\xi_1)$ is facilitated by letting ξ become very large, under which circumstance

$$R \longrightarrow F(\xi - \xi_1 \cos\theta), \quad \cos\Theta \longrightarrow \eta$$

and the left side of (2.35) approaches

$$\frac{k\,e^{ic\xi}}{c\xi} \cdot e^{-ic\xi_1 \cos\theta} \sin\theta \cdot$$

Furthermore, as specified in the preceding section,

$$R^{(3)}_{1n}(c\xi) \longrightarrow (-i)^{n+1} \frac{e^{ic\xi}}{c\xi},$$

and using these limits in (2.35) gives the form

$$e^{-ic\xi_1 \cos\theta} \sin\theta = \frac{1}{k} \sum_{n=0}^{\infty} \alpha_n(\xi_1) \cdot (-i)^{n+1} S_{1n}(c, \cos\theta). \tag{2.36}$$

Differentiating the well-known expansion

$$e^{-ic\xi_1 \cos\theta} = \sum_{n=0}^{\infty} (-i)^n (2n+1) j_n(c\xi_1) P_n(\cos\theta)$$

with respect to θ and using the fact that $\frac{\partial}{\partial\theta} P_n(\cos\theta) = P^1_n(\cos\theta)$ we can write

$$e^{-ic\xi_1 \cos\theta} \sin\theta = \frac{1}{ic\xi_1} \sum_{n=0}^{\infty} (-i)^n (2n+1) j_n(c\xi_1) P^1_n(\cos\theta)$$

$$= \frac{1}{k} \sum_{n=0}^{\infty} \alpha_n(\xi_1)(-i)^{n+1} S_{1n}(\cos\theta) \tag{2.37}$$

Multiplying this equation by $S_{1r}(\cos\theta)\sin\theta$ and integrating from 0 to π gives

$$\alpha_r(\xi_1)(-i)^r N_{1r} = \frac{k}{c\xi_1} \sum_{n=0}^{\infty} (-i)^n (2n+1) j_n(c\xi_1) \int_0^{\pi} P^1_n(\cos\theta) S_{1r}(\cos\theta) \sin\theta \, d\theta .$$

The integral on the right is easily dealt with by means of the expansion (2.7) for the angular function S_{1r}, and the result is

$$\alpha_r(\xi_1)(-i)^r N_{1r} = \frac{-2ik}{c\xi_1} \sum_{n=o,1}^{\infty}{}' (-i)^n \frac{(n+2)!}{n!} d_n^{1r} j_{n+1}(c\xi_1).$$

The sum on the right is now precisely equivalent to that appearing in the expansion (2.13) of the radial function $R_{mn}^{(1)}(c\xi_1)$ for $m = 1$, and the final result is therefore

$$\alpha_r(\xi_1) = \frac{-2(-i)^{r+1} k R_{1r}^{(1)}(c\xi_1)}{N_{1r}\rho_{1r}\sqrt{\xi_1^2 - 1}}$$

so that the desired expansion of H_ϕ is

$$H_\phi = -\frac{\omega i k^2 p}{2\pi\sqrt{\xi_1^2 - 1}} \sum_{n=0}^{\infty} \frac{(-i)^n}{\rho_{1n} N_{1n}} R_{1n}^{(1)}(c, \xi_1) R_{1n}^{(3)}(c, \xi) S_{1n}(c, \eta) \tag{2.39}$$

and the boundary condition (2.33) is applied to the total field, i.e. the sum of (2.38) and (2.39), with the radial functions interchanged in the latter, yielding at once

$$A_n(\xi_1) = \frac{-(-i)^n R_{1n}^{(3)}(c, \xi_1)}{\rho_{1n} N_{1n}} \left[\frac{\frac{\partial}{\partial\xi}\left(\sqrt{\xi^2 - 1}\, R_{1n}^{(1)}(c, \xi)\right)}{\frac{\partial}{\partial\xi}\left(\sqrt{\xi^2 - 1}\, R_{1n}^{(3)}(c, \xi)\right)} \right]_{\xi = \xi_0} \tag{2.40}$$

which, in conjunction with (2.39), gives the scattered field of the conducting spheroid excited by an electric dipole located in its axis of symmetry.

If the dipole is located in the surface of the spheroid, i.e. at the pole, then $\xi_1 = \xi_0$ and the expression for the Wronskian of the radial functions of first and

third types, which is i times the quantity given in (2.16), reduces the expression for the total field to

$$H_{\phi}^{T} = \frac{-\omega k^2 p}{2\pi F(\xi_o^2 - 1)} \sum_{n=0}^{\infty} \frac{\overline{(-i)^n R_{1n}^{(3)}(c,\xi) S_{1n}(c,\eta)}}{\rho_{1n} N_{1n} \left[\frac{\partial}{\partial \xi} \left(\sqrt{\xi^2 - 1}\; R_{1n}^{(3)}(c,\xi) \right) \right]_{\xi_o}} \tag{2.41}$$

The far zone radiation pattern is obtained as usual by inserting the asymptotic form of the radial function in the numerator of the above and dividing by the quantity e^{ikr}/r.

We close this section with some general remarks on the relations between scalar problems and axially symmetric vector problems involving the spheroid. In Kleinman and Senior (1963) it was shown how the vector solution for an infinite cone excited by a radial electric or magnetic dipole can be obtained by applying a vector operator to the solution of a physically meaningful scalar problem involving a point source and a simple (Dirichlet or Neumann) boundary condition satisfied by the total field on the conical surface. The problem is formulated there in terms of a pair of Debye potentials, which are independent solutions of the scalar wave equation, and the result just stated derives from the fact that in the particular coordinate system appropriate to the cone problem, the electromagnetic boundary condition can be satisfied if one of these Debye potentials vanishes identically and the other satisfies one of the above-named scalar boundary conditions on the cone.

Unfortunately in the spheroidal coordinate system this situation does not prevail. The general electromagnetic field can still be represented in terms of a pair of Debye potentials, but even if the corresponding scalar source is located in the axis of symmetry, the resulting expression for the tangential electric field component E_η contains one or the other of the potentials as well as its derivatives with respect to both ξ and η, so that no simple scalar boundary condition on either potential can make this component vanish. Thus it appears that, while the axially symmetric vector problem can still be solved in terms of a single scalar quantity, the corresponding scalar boundary value problem cannot be reduced directly to one of the standard forms previously derived, and probably has no physical interest in and of itself.

2.3 VECTOR PROBLEMS

In the preceding section it was shown how the solutions of certain electromagnetic problems involving spheroids could be obtained directly in terms of a single quantity which satisfies the wave equation and certain boundary conditions of a rather complicated form. The requirement of complete axial symmetry stipulated there is of course a stringent one, and rules out the important cases of arbitrary or transverse dipole sources, as well as the limiting case of a plane wave. Consideration of the latter, which is our next objective, requires a much more elaborate analytical apparatus, which we proceed to develop briefly.

In the preceding report of this series (Kleinman and Senior, 1963) a general formulation was given for the solution of an electromagnetic scattering problem in terms of a pair of Debye potentials or their associated Hertz vectors. This formulation could be applied to the spheroid problem, and the solution could presumably be carried out in some manner though not in the same sense that solutions to the cone and sphere problems have been. That is to say, the solution would not be obtained in closed form or even in terms of explicit expressions for the coefficients in an infinite series. The difficulty which arises is primarily concerned with the boundary conditions, and in order to bring this out more clearly, and also to follow existing literature on the problem, we present here a somewhat different (though essentially equivalent) formulation in terms of a set of vector wave functions analogous to those developed for the sphere problem by Hansen (cf. Stratton, 1941, p. 393).

The construction of these functions is perhaps best motivated by a brief discussion of the import of the term separability as applied to a vector problem. If a general vector solution of the wave equation.

$$\nabla^2 \underline{F}+k^2 \underline{F} = \nabla \nabla \cdot \underline{F} - \nabla\wedge\nabla\wedge\underline{F} + k^2 \underline{F} = 0 \qquad (2.42)$$

is resolved into components parallel to the coordinate axes, three scalar partial differential equations for the components result, each of which, except in rectangular Cartesian coordinates, involves more than one component, so that the

simultaneous solution of the system is prohibitively difficult. As pointed out in the preceding section, if the field is axially symmetric, the system degenerates for a suitable coordinate system, and the solution is easily found in terms of a single scalar wave function or potential. In the absence of such symmetry a more subtle resolution of the vector function inquestion is required. For most physical problems of the sort considered here it is advantageous to split the vector into two parts, one of which is the gradient of a scalar function and is called the longitudinal component, and the other of which is the curl of a vector and is called transverse. The scalar functions involved must then be solutions of the scalar wave equation and must satisfy boundary conditions which, at least in a system where this equation separates, are easily determined from the original vector ones. Thus, we write the longitudinal component as $\underline{L} = \nabla \phi$, where ϕ is a solution of the scalar wave equation. Being a gradient, however, the longitudinal vector component will in general have non-zero divergence, and accordingly will not be suitable for representation of a source-free electromagnetic field, so that our primary interest here is in the transverse component, which is divergence-free by virtue of its definition as a curl. This condition ensures that only two independent scalars are required to specify the vector quantity completely, and these should be chosen in such a way as to facilitate the satisfaction of the boundary conditions. In general it would be desirable to resolve the transverse field into two component solutions, one of wnich is tangential

to the scattering surface and the other normal to it. Unfortunately this is not possible in most coordinate systems, but for certain ones of importance namely those in which one of the scale factors is unity and the ratio of the other two is independent of the coordinate corresponding to the first, something approaching this objective can be achieved (cf. Morse and Feshbach, 1953 p. 1764). The tangential component is expressed as the vector

$$\underline{M} = \nabla\wedge\left(\hat{a}_1\,\omega(\xi_1)\,\psi\right),$$

where ξ_1 is the variable whose scale factor is unity, a_1 is the corresponding unit coordinate vector, $\omega(\xi_1)$ is such that $\frac{d^2\omega}{d\xi_1^2} = 0$, and ψ is a solution of the scalar wave equation. The third component cannot always be constructed normal to the first coordinate surface, but at least its curl can be made tangential to it if the vector function is defined as

$$\underline{N} = \frac{1}{k}\nabla\wedge\nabla\wedge\left(\hat{a}_1\,\omega\Omega\right)$$

with ω as before and Ω a solution of the scalar wave equation (which may or may not be identical to ϕ or ψ, as suits our purposes). The possibility of resolving a general vector solution into three components as described above, where the scalar quantities involved separate in the usual way, is perhaps the most practical definition of separability of a vector equation.

For the spherical coordinate system, this process has been carried out completely, (cf. Stratton, 1941), and one application is the well-known solution for

electromagnetic scattering by a sphere, in which explicit expressions are obtained for the coefficients in an expansion of the field in series of the $\underline{M}$ and $\underline{N}$ vectors over the indices of the common set of scalar solutions from which they are formed. In the spheroidal coordinate system, however, the vector wave equation is not completely separable in the above sense. The scale factors are such that the transverse field cannot be resolved into components which permit the satisfaction of boundary conditions by the individual members of the series, and the best that can be done is to obtain an infinite system of equations for the infinite set of coefficients, which can be solved approximately by truncation.

In the above forms, the vector $\hat{a}_1$ was specified as a unit coordinate vector. Actually solutions to (2.42) are obtained if $\hat{a}_1$ is any constant vector, or even the radius vector $\underline{r}$. This permits considerable freedom in the choice of a particular set of vector functions for a given problem, and the determination of the optimum choice, i.e. the set which minimizes the labor or complication, is not easy. To the best of our knowledge, the question has not been absolutely settled for the spheroid problem, and we limit the present account to an outline of the solution which exists in the literature and which was given by Schultz (1950). This assumes a plane electromagnetic wave incident on a perfectly conducting spheroid and propagating in the direction of the major axis. The generalization to the case of arbitrary incident direction adds little of analytical interest.

If $\underline{a}$ is an arbitrary constant vector and $\psi_{\substack{e\\o}mn}^{(i)}$ is a separated solution of the scalar wave equation, where the index i = 1, 2, 3, 4 denotes the type of the radial function involved (the angle function is always of the first type), then application of the forms given above yields the various sets of vector functions

$$\underline{L}_{\substack{e\\o}mn}^{(i)} = \nabla \psi_{\substack{e\\o}mn}^{(i)}$$

$$\underline{M}_{\substack{e\\o}mn}^{a(i)} = \nabla \psi_{\substack{e\\o}mn}^{(i)} \wedge \underline{a} \tag{2.43}$$

$$\underline{N}_{\substack{e\\o}mn}^{a(i)} = \frac{1}{k} \nabla \wedge \underline{M}_{\substack{e\\o}mn}^{a(i)}$$

where the e and o subscripts denote even and odd ϕ -dependence, as before. In these we must first specify the vector $\underline{a}$ and then select whichever sets of functions are best suited for representing the fields we are dealing with. As noted previously the $\underline{L}$ functions will be of no use for the present problem since their divergence does not vanish. Actually, in contrast to the classical sphere solution, the spheroid solution of Schultz does not employ the $\underline{N}$ vectors either. Instead, three distinct sets of $\underline{M}$ vectors are generated by substituting for $\underline{a}$ the three Cartesian coordinate vectors $\hat{i}_x$, $\hat{i}_y$, $\hat{i}_z$. The completeness of these sets follows, by a simple argument, from that of the set of scalar wave functions (cf. Siegel et al, 1953) so that the possibility of expanding any solution to the given boundary value problem in a convergent series of these functions is assured. In the particular case considered

here only certain of the $\underline{M}$ vectors are required, and in the interest of economy we will list below only those to be used. Detailed expressions for the rest appear in Flammer (1957).

If the incident plane wave is assumed to propagate in the negative z-direction with electric and magnetic vectors in the positive y- and x-directions respectively, then a brief examination of the forms given above indicates that its electric vector should have an expansion of the form

$$\underline{E}^i = E^i e^{-ikz}\hat{i}_y = -\frac{i}{k} E^i \sum_{n=o}^{\infty} A_n \underline{M}_{eon}^{x(1)} \qquad (2.44)$$

(The choice of the even functions here is obvious with the assumed polarization, and since it develops that only even functions are needed throughout the solution, we can drop the e subscript with the understanding that all wave functions are even in ϕ unless otherwise stated.) From the expansion of the exponential in spheroidal wave functions (equation 2.36), and the definition (2.43) of the $\underline{M}$ functions, it is easily determined that

$$A_n = 2\, i^n S_{on}(1)/N_{on}. \qquad (2.45)$$

If the spheroid $\xi = \xi_o$ is perfectly conducting and if the total electric field is represented as

$$\underline{E} = \underline{E}^i + \underline{E}^s$$

where $\underline{E}^s$ represents the scattered field, then the boundary condition at the surface can be written

$$\hat{i}_\xi \wedge \underline{E}\Big|_{\xi=\xi_o} = 0 \text{ or } E^i_\eta + E^s_\eta\Big|_{\xi=\xi_o} = 0 = E^i_\phi + E^s_\phi\Big|_{\xi=\xi_o} . \tag{2.46}$$

Now since the $\underline{M}$ vectors form a complete set, we can assume an expansion of the scattered field of the form

$$\underline{E}^s = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \left(A^x_{mn} \underline{M}^{x(3)}_{mn} + A^y_{mn} \underline{M}^{y(3)}_{mn} + A^z_{mn} \underline{M}^{z(3)}_{mn} \right) \tag{2.47}$$

The third type of function is dictated by the radiation condition at infinity. Each of the three sets of $\underline{M}$ vectors has its own characteristic ϕ-dependence which is expressed in terms of ordinary trigonometric functions, and since the boundary conditions (2.46) must hold for all ϕ, the orthogonality of these functions may be invoked to reduce drastically the variety of $\underline{M}$ vectors appearing in the series (2.47) for the scattered wave. It develops immediately on substituting the expansion of the $\underline{M}$ vectors in (2.47) and applying (2.46) that the only sets of vectors whose coefficients do not vanish are $\underline{M}^{x(3)}_{on}$, and $\underline{M}^{z(3)}_{1n}$, and we can rewrite (2.47) accordingly as

$$\underline{E}^s = \sum_{n=0}^{\infty} \left(A^x_n \underline{M}^{x(3)}_{on} + A^z_n \underline{M}^{z(3)}_{1n} \right) . \tag{2.48}$$

The essential problem now facing us is the determination of the coefficients A^x_n and A^z_n in (2.48), to which end we must make whatever use we can of the

boundary conditions. In order to see the exact nature of the difficulties we first express the $\underline{M}$ vectors in terms of the spheroidal coordinates using (2.43) and the transformation of coordinate vectors given at the beginning of this chapter. Thus we have

$$\underline{M}^{x\,(1)}_{on} = \frac{\hat{i}_\xi}{F}\left\{\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\;\frac{dS_{on}}{d\eta}\;R^{(1)}_{on}\;\sin\phi\right\} \tag{2.49}$$

$$+\frac{\hat{i}_\eta}{F}\left\{-\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,S_{on}\,\frac{d}{d\xi}\,R^{(1)}_{on}\;\sin\phi\right\}$$

$$+\frac{\hat{i}_\phi}{F}\left\{\frac{\eta(\xi^2-1)}{(\xi^2-\eta^2)}\,S_{on}\,\frac{d}{d\xi}\,R^{(1)}_{on}\,\cos\phi+\frac{\xi(1-\eta^2)}{(\xi^2-\eta^2)}\,\frac{d}{d\eta}\,S_{on}R^{(1)}_{on}\,\cos\phi\right\}$$

$$\underline{M}^{z\,(3)}_{1n} = \frac{\hat{i}_\xi}{F}\left\{\frac{-\xi}{\sqrt{(\xi^2-1)(\xi^2-\eta^2)}}\,S_{1n}R^{(3)}_{1n}\,\sin\phi\right\}+\frac{\hat{i}_\eta}{F}\left\{\frac{\eta}{\sqrt{(1-\eta^2)(\xi^2-\eta^2)}}\,S_{1n}R^{(3)}_{1n}\,\sin\phi\right\}$$

$$+\frac{\hat{i}_\phi}{F}\left\{\frac{-\xi\sqrt{(\xi^2-1)(1-\eta^2)}}{(\xi^2-\eta^2)}\,S_{1n}\,\frac{d}{d\xi}\,R^{(3)}_{1n}\,\cos\phi+\frac{\eta\sqrt{(\xi^2-1)(1-\eta^2)}}{(\xi^2-\eta^2)}\,\frac{d}{d\eta}S_{1n}R^{(3)}_{1n}\cos\phi\right\}$$

The third set, $\underline{M}^{x(3)}_{on}$, is of course identical to $\underline{M}^{x(1)}_{on}$ except that the radial function $R^{(3)}_{on}$ appears throughout in place of $R^{(1)}_{on}$. Substitution of these in the field expansions (2.44) and (2.48), followed by application of the boundary equations (2.46) yields the two equations

$$\frac{i}{k}E^1\sum_{n=o}^{\infty}A_n\sqrt{(\xi_o^2-1)}\,S_{on}(\eta)\left[\frac{d}{d\xi}\,R^{(1)}_{on}(\xi)\right]_{\xi_o} = \sum_{n=o}^{\infty}A^x_n\sqrt{\xi_o^2-1}\,S_{on}(\eta)\left[\frac{d}{d\xi}R^{(3)}_{on}(\xi)\right]_{\xi_o}$$

$$-\sum_{n=o}^{\infty}A^z_n\,\frac{\eta}{\sqrt{1-\eta^2}}\,S_{1n}(\eta)\,R^{(3)}_{1n}(\xi_o) \tag{2.50}$$

$$\frac{iE^i}{k}\sum_{n=0}^{\infty} A_n\left\{(\xi_0^2-1)\eta S_{on}(\eta)\left[\frac{d}{d\xi}R_{on}^{(1)}(\xi)\right]_{\xi_0}+\xi_0(1-\eta^2)\frac{d}{d\eta}S_{on}(\eta)R_{on}^{(1)}(\xi_0)\right\}$$

$$=\sum_{n=0}^{\infty} A_n^x\left\{\eta(\xi_0^2-1)S_{on}(\eta)\left[\frac{d}{d\xi}R_{on}^{(3)}(\xi)\right]_{\xi_0}+\xi_0(1-\eta^2)\frac{d}{d\eta}S_{on}(\eta)R_{on}^{(3)}(\xi_0)\right\} \tag{2.51}$$

$$-\sum_{n=0}^{\infty} A_n^z\left\{\xi_0\sqrt{(\xi_0^2-1)(1-\eta^2)}\,S_{1n}(\eta)\left[\frac{d}{d\xi}R_{1n}^{(3)}(\xi)\right]_{\xi_0}-\eta\sqrt{(\xi_0^2-1)(1-\eta^2)}\,\frac{d}{d\eta}S_{1n}(\eta)R_{1n}^{(3)}(\xi_0)\right\}.$$

The essential complication of the spheroidal geometry now becomes apparent. In the corresponding equations for the sphere, the angular dependence is such that the orthogonality of the angle functions can be applied directly to give two simple expressions relating the known coefficient A_n and the two unknown ones A_n^x, A_n^z with the same index n, and the scattered field is thus expressed in terms of the series (2.48), all of whose coefficients are easily written down. For the spheroid, however, this is not possible. The appearance of the scale factors and of angle functions with two different values of the index m makes it impossible to relate the known and unknown coefficients with the same, or even with a finite number of distinct indices n. Since there are no recurrence relations for the spheroidal functions, there seems to be no easy way around this difficulty, and the best one can do is to obtain an infinite system of simultaneous equations for the unknowns and resort to a large scale computing program for its solution.

Such a system was constructed by Schultz in the following manner. Equations (2.50) and (2.51) are multiplied by a function $S_{or}(\eta)$, with r ranging from 0 to ∞, and integrated over the range $-1 \leq \eta \leq 1$. The result is a doubly infinite system of equations in the unknown coefficients A_n^x, A_n^z, which can be written

$$\sum_{n=o}^{\infty}\left(C_{rn} A_n^x + D_{rn} A_n^z\right) = E^i a \sum_{n=o}^{\infty} B_{rn} \tag{2.52}$$

$$\sum_{n=o}^{\infty}\left(V_{rn} A_n^x + W_{rn} A_n^z\right) = E^i a \sum_{n=o}^{\infty} U_{rn} \tag{2.53}$$

where r = 0, 1, 2 . . . ∞, and the known quantities are

$$B_{rn} \equiv \frac{iA_n}{c\xi_o} \sqrt{\xi_o^2 - 1}\; \frac{dR_{on}^{(1)}}{d\xi}\bigg|_{\xi_o} \int_{-1}^{1} S_{on} S_{or}\, d\eta$$

$$C_{rn} \equiv \sqrt{\xi^2 - 1}\; \frac{dR_{on}^{(3)}}{d\xi}\bigg|_{\xi_o} \int_{-1}^{1} S_{on} S_{or}\, d\eta \tag{2.54}$$

$$D_{rn} \equiv -R_{1n}^{(3)}(\xi_o) \int_{-1}^{1} \frac{\eta}{\sqrt{1-\eta^2}}\, S_{1n} S_{or}\, d\eta$$

$$U_{rn} \equiv \frac{iA_n}{c\xi_o}\left\{(\xi_o^2 - 1)\frac{d}{d\xi} R_{on}^{(1)}\bigg|_{\xi_o} \int_{-1}^{1} \eta S_{on} S_{or}\, d\eta + \xi_o R_{on}^{(1)}(\xi_o) \int_{-1}^{1} (1-\eta^2)\frac{dS_{on}}{d\eta} S_{or}\, d\eta\right\}$$

$$V_{rn} \equiv (\xi_o^2 - 1) \left. \frac{dR_{on}^{(3)}}{d\xi} \right|_{\xi_o} \int_{-1}^{1} \eta\, S_{on} S_{or}\, d\eta + \xi_o R_{on}^{(3)} \int_{-1}^{1} (1-\eta^2) \frac{dS_{on}}{d\eta} S_{or}\, d\eta$$

$$W_{rn} \equiv -\xi_o \sqrt{\xi_o^2 - 1} \left. \frac{dR_{1n}^{(3)}}{d\xi} \right|_{\xi_o} \int_{-1}^{1} \sqrt{1-\eta^2}\, S_{1n} S_{or}\, d\eta + \sqrt{\xi_o^2 - 1}\, R_{1n}^{(3)}(\xi_o) \int_{-1}^{1} \eta \sqrt{1-\eta^2} \frac{dS_{1n}}{d\eta} S_{or}\, d\eta$$

With the exception $\int_{-1}^{1} S_{on} S_{or}\, d\eta$, which is of course equal to $N_{or}\ \delta_{rn}$, the above integrals cannot be evaluated in closed form. They can, however, be expressed in series of spheroidal coefficients by simply expanding each S_{mn} in series of Legendre functions and using the orthogonality properties of the latter. The actual expressions are given in Chapter IV.

The convergence of the above system of equations could presumably be demonstrated rigorously by straightforward methods, but this seems hardly worthwhile at this point in view of the reasonableness of the results and the simple physical arguments which support it. As noted previously, the system can be solved approximately by truncation, i.e., by taking only the first N equations of each set and solving for the first N pairs of unknowns. The number N depends, of course, on the size (and eccentricity) of the body and on the accuracy desired. The fact that over half the terms in the system vanish identically is of some small benefit in the computation task, though this gain is rather overbalanced by the circumstance that

the remaining ones are complex. Details of extensive computations based on this solution and done on a large scale digital machine are given by Siegel et al (1953) and Ritter (1956).

An expression for the scattered field in the far zone is of primary interest in radar problems and can be obtained from (2.48) by substituting the asymptotic forms (2.15) of the radial functions into the expressions (2.49) for the $\underline{M}$ vectors. Thus at large distances r from the spheroid*, the scattered field is

$$\underline{E}^s \cong \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} i^n \left\{ \hat{i}_\eta \left[-A_n^x S_{on}(\eta) \sin\phi \right] \right.$$

$$\left. + \hat{i}_\phi \left[A_n^x \eta S_{on}(\eta) - i A_n^z \sqrt{1-\eta^2} S_{1n}(\eta) \right] \cos\phi \right\} . \tag{2.55}$$

In the generalization of this solution to the case of arbitrary direction of incidence (Reitlinger, 1957) the field must be expanded in a double series, with the index m running from zero to infinity. The proper choice of the $\underline{M}$ vectors becomes even more difficult, and the matching of terms in the boundary equations by means of the ϕ - dependence is not so trivial. The η - dependence is also more complicated, so that the matching procedure used above produces not only the η integrals encountered there but also numerous others of similar form, and leads to an infinite

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*The criteria for the validity of these forms are that $(c\xi)^2 >> \lambda_{mn}$ and $(c\xi)^2(\xi^2-1)$ m^2.

set of infinite systems of equations for the expansion coefficients. The mere transcription of these is a rather formidable task, and it is hoped that the reader will never be faced with the necessity of using them.

Another extension of the wave-function solution might be discussed briefly at this point. This deals with the case of a dielectric scatterer. A radiation problem of this type was treated by Weeks (1958). The case considered is that of a homogeneous dielectric spheroid covered by a spheroidal shell of another dielectric and excited by a transverse slot at each of several locations. This type of excitation is strictly outside the range of the present report, and the details of the solution will not be given here, but certain of the results are included in another section. The general technique, i.e., method of representation of the fields, is the same as in the case of the perfect conductor, but here, of course, the field interior to the body must also be considered. The problem of matching the fields at the boundaries is even further complicated by the fact that two different values of the wave number k are involved, and since this appears as a parameter in all the spheroidal functions, the orthogonality relations are further restricted. Instead of multiplying the boundary equations by a spheroidal function and then integrating, as in the solution of Schultz, it is necessary to expand each angle function appearing there is series of Legendre functions and then employ the orthogonality properties

of the latter*. The result is again an infinite system of equations in the infinite set of unknown expansion coefficients for the radiated field, numerical treatment of which requires a large scale computing facility. The convergence properties of this system would presumably be similar, though not necessarily identical, to those of the system constructed by Schultz. This question has not been thoroughly investigated, however.

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* A variation on this procedure has been given more recently by Yeh (1963) who published a formal solution of the same problem, presumably being unaware of the existence of Weeks' earlier and more extensive work. In this paper the angular functions pertaining to one medium are expanded directly in terms of those of the other, the coefficients being expressed as series involving the two sets of d_r^{mn}. The two solutions are essentially equivalent, and it is not immediately clear which form is preferable. In an earlier report by Johnson (1955) the problem of the dielectric spheroid was attacked by means of a set of approximate vector wave functions, which satisfy the wave equation only in the far-zone limit. The procedure used in determining the expansion coefficients of the reflected and transmitted waves, namely that of applying the boundary conditions to the various series in terms of these functions, would seem to be of very doubtful validity except in the case of a nearly spherical scatterer. The solution should thus perhaps be classified with the eccentricity-restricted approximations, but the region of validity might be difficult to determine.

III

APPROXIMATE SOLUTIONS

3.1 FREQUENCY-RESTRICTED APPROXIMATIONS

The wave-function solutions detailed in the preceding chapter have been called exact because it is theoretically possible to carry them out to any desired degree of accuracy. In practice, however, this presents serious difficulties. So far little has been said about the convergence of the series involved, but it is not hard to show that this becomes slower as the frequency increases, and as in the case of the sphere, for a given frequency, the summation indices must reach a value considerably in excess of ka in order to yield any reasonable accuracy. For the scalar problem with symmetric excitation, the exact solution has been carried out to a high degree of accuracy for certain spheroids at frequencies ranging up to a value of $ka \simeq 4$. For the vector case, however, even with plane wave incidence in the axis of symmetry the few existing computations are accurate only out to $ka \simeq 3$, and for other directions of incidence, no computations have even been attempted, to the best of our knowledge. The need for approximate solutions which offer reasonable accuracy at tolerable expense is thus obvious, and several of these have been developed. None, of course, is useful over the entire ranges of interest in all the parameters, and the natural basis of classification is the parameter or parameters restricted and the ranges of validity.

Perhaps the most important parameters in this respect are wavelength (relative to characteristic dimension) and eccentricity. The material properties of the scatterer may also require consideration, and several investigators have developed solutions based on the perturbations of these properties with reference to the surrounding medium; these will be discussed presently. By far the greater part of the existing approximate theory however, depends primarily on the aforementioned geometric parameters. We will deal first with the matter of wavelength or frequency, and begin at the low-frequency end of the spectrum, which is generally referred to as the Rayleigh region, after the author who provided the first systematic treatment of low-frequency scattering (Rayleigh, 1897).

3.1.1 Low Frequency Approximations

When the wavelength of the energy incident on a body is large compared to the characteristic dimension of the body, then ka is small, and this immediately suggests a series representation for the scattered field in powers of this quantity. This series is usually referred to as the Rayleigh series, despite the fact that Rayleigh's original contribution only yielded the first term. In general it affords the easiest and most universally practical way of obtaining the scattered field of an object in the region of the spectrum where the first few terms provide sufficient accuracy. Its use at higher frequencies is limited absolutely by the finite radius of convergence of the series, and practically, of course, by the difficulty of obtaining the coefficients for the higher order terms, especially in the vector case. Besides

the obvious advantage that once the coefficients are known, the scattered field is immediately obtainable for any wavelength within the region of effective convergence, this solution has the further merit that the cases of arbitrary incident direction and shape and material parameters present no essential difficulty. It is thus the most general solution known, the only important restriction being that of large wavelength.

To date, the coefficients in the Rayleigh series have not been obtained explicitly beyond the third, or in certain cases the fourth (non-vanishing) term in the far-field expansion. There are perhaps two principal reasons for this, the first being the fact that the majority of the methods presently available either break down completely or have not been developed sufficiently to yield more terms, or would involve a prohibitive amount of algebra, and the second, the fact that the limited region and non-uniform manner of convergence of the series (cf. Senior, 1961) seriously restricts the advantages to be gained. The first extensions of Rayleigh's work were produced almost simultaneously by Tai (1952), and Stevenson (1953 a), who developed quite distinct methods for obtaining the next non-vanishing term (which is proportional to k^4 in the far field series*, the term in k^3 vanishing

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*The term "far field" is used here to denote the coefficient of $\frac{e^{ikr}}{r}$ in the scattered field expression, in contrast to the treatment below, where the coefficient of $\frac{e^{ikr}}{kr}$ is considered.

identically for a body with a center of symmetry). The two derivations have been discussed and compared by Justice (1956) and will be outlined below. The third non-vanishing term, proportional to k^5, was derived by Senior (1960) for the scalar problem with nose-on incidence and either Dirichlet or Neumann boundary conditions, and by the same author (1964) for the vector problem with a perfectly conducting spheroid. (In the latter work only the coefficients in the power series expansions of the wave-function coefficients are given explicitly, but from these the Rayleigh coefficients are easily obtainable.) Additional power series coefficients in the scalar problem are also given by Senior (1961), but these are not sufficient to extend the Rayleigh series for the far field beyond the term mentioned above.*

For the scalar problem, the Rayleigh series is obtainable in a straightforward manner from the exact solution by simply substituting power series expansions for all quantities which depend on k and then rearranging terms and collecting coefficients of like powers of k. Logically, of course, this procedure might be termed reverse, presupposing, as it does, a knowledge of the exact solution and of the functions in terms of which the latter is naturally expressed, and yielding only an approximate form with a more restricted range of validity. However, the method is easily justified on practical grounds, since in the cases where it

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*It should be noted that the statement of Sleator (1960)(also in Crispin et al, 1963), that the coefficient of k^5 vanishes for the hard (scalar) spheroid, is in error, as is the value given for the coefficient of k^6. The corrected curve of cross section vs ka is shown in Fig. 14.

is applicable it offers the easiest access to the above-mentioned advantages of the power series representation for the scattered field. Accordingly we defer any discussion of Rayleigh's original derivation until the consideration of the vector problem where the exact solution is not known in such generality and the advantages of Rayleigh's methods are thus more evident.

The exact solution for the scalar problem with incident plane wave and linear homogeneous boundary condition was given in equation (2 24). For the sake of convenience we now restrict the incident direction to the axis of symmetry and consider the limiting cases of the hard and soft spheroids ($\alpha/\beta = 0, \infty$, respectively) separately, with the observation that the general impedance solution can be easily reconstructed from these. Following Senior (1960) we can write the far field amplitude for the soft body as

$$f_s(\eta) = 2i \sum_{n=0}^{\infty} \frac{S_{on}(c,-1)}{N_{on}} \frac{R_{on}^{(1)}\left(c,\xi_o\right)}{R_{on}^{(3)}\left(c,\xi_o\right)} S_{on}(c,\eta) \tag{3.1}$$

where $f_s(\eta)$ is defined by the relation that if V^s is the scattered field in the far zone, then

$$V^s = f_s(\eta)\frac{e^{ikR}}{kR} \,.$$

(Note that this definition of f differs slightly from that used elsewhere by virtue of the k in the denominator of the expression for V^s.) When the spheroidal functions

appearing in (3.1) are expanded in Legendre functions (cf. equations (2.7) and (2.17)) their dependence on c is contained entirely in the coefficients, which are either the d_r^{mn} of equation (2.7) or are directly obtainable from them, and by using the recurrence relation and normalizing equation which serve to define the d_r^{mn}, the expansion of these in powers of c is easily accomplished, at least to a reasonable number of terms, (see Senior (1960) for details.) If we then write the Rayleigh series for f_s as

$$f_s(\eta) = -c \sum_{n=o}^{\infty} u_n(\eta)(-ic)^n, \tag{3.2}$$

the functions $u_n(\eta)$ are found by collecting the coefficients of like powers of c in the completely expanded form of f_s. The expression analogous to (3.1) for the hard spheroid is identical to it except that the radial functions are replaced by their derivatives with respect to ξ, so that the requisite expansions of these are in terms of the derivatives of the Legendre functions. The same procedure described above applies here, and the functions $v_n(\eta)$ in the Rayleigh expansion for the far field amplitude

$$f_h(\eta) = -c \sum_{n=o}^{\infty} v_n(\eta)(-i\,c)^n \tag{3.3}$$

are thus determined. There are substantial differences in the two results, in that the functions v_o, v_1, and v_3 for the hard body vanish identically, whereas none of

the first six u functions in the soft case vanishes. The complete forms for the two sets of functions are listed for $n = 0 ------ 5$ in Sec. 4.1.

The radius and manner of convergence of the series (3.2) and (3.3) are discussed at length by Senior (1961), and the details are too involved to be treated fully in the present work. In general the radius of convergence can be determined by considering the coefficients in the wave-function (exact) solution as functions of the complex variable $\rho \equiv ka$ and locating the pole of least amplitude among all the poles of all the coefficients. For the sphere, this minimum amplitude is unity for both the soft and hard cases, and the Rayleigh series accordingly converges only out to the value $ka = 1$. If the sphere is elongated in the direction of incidence, so that a is the semi-major axis of the resulting prolate spheroid then the radius of convergence increases for both hard and soft bodies, though in a different manner for each, approaching the value $ka \cong 4.1$ in both cases as the spheroid becomes an infinitely thin rod. For all values of the eccentricity between zero and one, the radius of convergence for the hard body exceeds that for the soft, the greatest difference occurring when the axis ratio is around .02, where its magnitude is approximately 2.0. The above discussion applies only for the Dirichlet and Neumann boundary conditions. For the general linear homogeneous boundary condition (2.22) the situation is much more complicated and the convergence radius can be expected to decrease as the ratio α/β departs from the values 0 or ∞.

For the vector or electromagnetic problem, the derivation of the Rayleigh series is predictably more involved, and several methods have been used for the determination of the first three terms in the power series for the scattered field. Two of these are the previously mentioned solutions of Stevenson (1953 a, b) and Tai (1952) which have since been elucidated and compared in a report by Justice (1956). To date no further terms have been derived for the vector case, nor has the convergence question been discussed adequately, and the predictable accuracy of the solution rests primarily on a comparison of particular results with those given by the exact solution, as presented in a later section of this report.

In view of the difficulties inherent in the derivation of the complete Rayleigh series, and as a matter of historical interest, it seems appropriate to discuss briefly Rayleigh's original derivation of the first term, which he accomplished by means of a quite general and remarkably simple line of argument. The derivation assumes the existence of a region where the distance from the scatterer is large compared to its dimensions but small compared to the wavelength of the incident field. Here the solution is basically that of a static problem, and once this is known, the field at a larger distance can be immediately found from the known properties of spherical harmonics. The method is essentially the same for both scalar and vector problems, and the material properties of the scatterer are easily taken into account. We consider here only the electromagnetic problem for a homogeneous

ellipsoid of permeability μ' and dielectric constant ϵ' in a medium of corresponding constants μ, ϵ struck by a plane wave propagating parallel to the major axis. In the interest of readability and consistency, we will modify Rayleigh's notation to agree with our previous usage wherever possible, and accordingly we denote the major axis by $2F\xi_0$ and let it coincide with the z-axis. The electric and magnetic vectors of the incident wave in the region exterior to the spheroid are then represented as

$$\begin{aligned} \underline{E}^i &= E^i e^{-ikz}\,\hat{i}_y \\ \underline{H}^i &= \sqrt{\frac{\epsilon}{\mu}}\,E^i e^{-ikz}\,\hat{i}_x \end{aligned} \tag{3.4}$$

where E^i, the amplitude of the incident wave, may be normalized to unity. In the neighborhood of the obstacle, under the assumption that ka is small so that $e^{-ikz} \simeq 1$, the incident electric and magnetic fields are derivable from two scalar potentials, i.e.

$$\begin{aligned} \underline{E}^i &= E^i \nabla \phi_e^i, \quad \phi_e^i = y \\ \underline{H}^i &= \sqrt{\frac{\epsilon}{\mu}}\,E^i \nabla \phi_m^i, \quad \phi_m^i = x \end{aligned} \tag{3.5}$$

Within this region the scattered field is also derived from potentials, which can in general be expanded in series of spherical harmonics in the form

$$\phi^s_{e,m} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} A^{e,m}_{nj} P_n^j(\cos\theta) e^{ij\phi} r^{-n-1} \quad .$$

Since there are no sources present, the term with $n = 0$ must vanish. Furthermore in the region of interest, $F\xi_0 << r << \lambda$, the terms with $n > 1$ are negligible, and the potentials reduce to

$$\phi^s_{e,m} = \sum_{j=-1}^{1} A^{e,m}_{1j} P^j_1(\cos\theta)\, e^{ij\phi} r^{-2},$$

which may be rewritten in terms of three new constants as

$$\phi^s_{e,m} = \left(A^{e,m}_x x + A^{e,m}_y y + A^{e,m}_z z\right)/r^3$$

If we consider these constants as the rectangular components of constant vectors $\underline{A}^{e,m}$, then the above potentials can be written

$$\phi^s_{e,m} = -\nabla \cdot (\underline{A}^{e,m}/r)$$

and the 'scattered' electric and magnetic fields in this intermediate range ($1/r >> 1/r^2$, $kr << 1$) become

$$\underline{E}^s = -E^i \nabla\left(\nabla \cdot \frac{\underline{A}^e}{r}\right)$$

$$\underline{H}^s = -\sqrt{\frac{\epsilon}{\mu}}\, E^i \nabla\left(\nabla \cdot \frac{\underline{A}^m}{r}\right) \quad . \tag{3.6}$$

These expressions are not adequate to represent the far field, for when kr is appreciable the magnetic potential contributes to the electric field and vice versa. To deal with this region we use a Hertz vector representation, noting that any field may be written in terms of electric and magnetic Hertz vectors as follows

(see, for example, Kleinman and Senior, 1963):

$$\underline{E}^S = \nabla\wedge\nabla\wedge\,\underline{\pi}_e + i\omega\mu\,\nabla\wedge\underline{\pi}_m = \nabla\,(\nabla\cdot\underline{\pi}_e) - \nabla^2\underline{\pi}_e + ik\sqrt{\frac{\mu}{\epsilon}}\;\nabla\wedge\underline{\pi}_m \tag{3.7}$$

$$\underline{H}^S = \nabla\wedge\nabla\wedge\,\underline{\pi}_m - i\omega\epsilon\,\nabla\wedge\underline{\pi}_e = \nabla(\nabla\cdot\underline{\pi}_m) - \nabla^2\underline{\pi}_m - ik\sqrt{\frac{\epsilon}{\mu}}\;\nabla\wedge\underline{\pi}_e\ .$$

If the vectors $\underline{\pi}_e$ and $\underline{\pi}_m$ are specified as dipoles located at the origin, then

$$\underline{\pi}_e = \frac{e^{ikr}}{r}\,\underline{C}_e, \qquad \underline{\pi}_m = \frac{e^{ikr}}{r}\,\underline{C}_m \tag{3.8}$$

where $\underline{C}_e$ and $\underline{C}_m$ are constant vectors (i e. dipole moments.) Furthermore in the range where $kr \simeq 0$, we have

$$\underline{\pi}_e \simeq \frac{\underline{C}_e}{r} \quad \text{and} \quad \underline{\pi}_m \simeq \frac{\underline{C}_m}{r} \tag{3.9}$$

so that (3.7) becomes

$$\underline{E}^S = \nabla(\nabla\cdot\underline{\pi}_e) \simeq \nabla\left(\nabla\cdot\frac{\underline{C}_e}{r}\right) \tag{3.10}$$

$$\underline{H}^S = \nabla\,(\nabla\cdot\underline{\pi}_m) \simeq \nabla\left(\nabla\cdot\frac{\underline{C}_m}{r}\right)\ .$$

Identifying (3.6) and (3.10) we obtain the expressions

$$\underline{C}_e = -E^i\,\underline{A}^e, \qquad \underline{C}_m = -\sqrt{\frac{\epsilon}{\mu}}\;E^i\,\underline{A}^m, \tag{3.11}$$

and the far field is then given by (3.6), (3.8) and (3.11). This approximation to the field is thus completely defined once the constant vectors $\underline{A}_e$ and $\underline{A}_m$ are specified. These are obtained by considering the static problem for the spheroid.

Rayleigh gives the solution to the static problem in the following form. If the impressed field potentials $\phi^{i}_{e,m}$ have the form

$$\phi^{i}_{e,m} = ux + vy + wz$$

for some constants u, v, w, then the vectors $\underline{A}^{e,m}$ have components

$$A_x^{e,m} = -\frac{uVK_{e,m}}{1+K_{e,m}L}, \quad A_y^{e,m} = -\frac{vVK_{e,m}}{1+K_{e,m}M}, \quad A_z^{e,m} = -\frac{wVK_{e,m}}{1+K_{e,m}N}$$

where V is the volume of the spheroid,

$$K_e = (\epsilon'/\epsilon - 1)/4\pi$$

$$K_m = (\mu'/\mu - 1)/4\pi$$

$$L = M = 2\pi\left\{\xi_o^2 - \frac{\xi_o(\xi_o^2 - 1)}{2}\log\frac{\xi_o+1}{\xi_o-1}\right\}$$

and

$$N = 4\pi(\xi_o^2 - 1)\left\{\frac{\xi_o}{2}\log\frac{\xi_o - 1}{\xi_o+1} - 1\right\}.$$

In the present case (3.5), $\phi^{i}_{e} = y$, i.e. u = w = 0, v = 1, and thus

$$A_x^e = A_z^e = 0, \quad A_y^e = -\frac{V(\epsilon' - \epsilon)}{4\pi\epsilon + 2\pi(\epsilon' - \epsilon)\left\{\xi_o^2 - \frac{\xi_o(\xi_o^2 - 1)}{2}\log\frac{\xi_o+1}{\xi_o-1}\right\}}$$

and $\phi^{i}_{m} = x$, i.e. v = w = 0, u = 1, so that

$$A_y^m = A_z^m = 0, \quad A_x^m = -\frac{V(\mu' - \mu)}{4\pi\mu + 2\pi(\mu' - \mu)\left\{\xi_o^2 - \frac{\xi_o(\xi_o^2 - 1)}{2}\log\frac{\xi_o+1}{\xi_o-1}\right\}}$$

In the case of perfect conductivity, $\epsilon' \to \infty$ and $\mu' \to 0$ and the expressions become

$$A_y^e \to \frac{-V}{2\pi\left\{\xi_o^2 - \frac{\xi_o(\xi_o^2-1)}{2}\log\frac{\xi_o+1}{\xi_o-1}\right\}}, \quad A_x^m \to \frac{V}{4\pi - 2\pi\left\{\xi_o^2 - \frac{\xi_o(\xi_o^2-1)}{2}\log\frac{\xi_o+1}{\xi_o-1}\right\}}. \tag{3.12}$$

With these constants thus defined, the scattered field of a plane wave incident along the axis of symmetry may be written explicitly as

$$\underline{E}^s = -E^i\left\{A_y^e\,\nabla\wedge\nabla\wedge\left(\frac{e^{ikr}}{r}\hat{i}_y\right) + ikA_x^m\,\nabla\wedge\left(\frac{e^{ikr}}{r}\hat{i}_x\right)\right\}$$

$$\underline{H}^s = -\sqrt{\frac{\epsilon}{\mu}}\,E^i\left\{A_x^m\,\nabla\wedge\nabla\wedge\left(\frac{e^{ikr}}{r}\hat{i}_x\right) - ikA_y^e\,\nabla\wedge\left(\frac{e^{ikr}}{r}\hat{i}_y\right)\right\} \tag{3.13}$$

or alternatively

$$\underline{E}^s = -E^i\left\{A_y^e\left(\nabla\frac{\partial}{\partial y}\frac{e^{ikr}}{r} + k^2\frac{e^{ikr}}{r}\hat{i}_y\right) + ikA_x^m\,\nabla\frac{e^{ikr}}{r}\wedge\hat{i}_x\right\}$$

$$\underline{H}^s = -\sqrt{\frac{\epsilon}{\mu}}\,E^i\left\{A_x^m\left(\nabla\frac{\partial}{\partial x}\frac{e^{ikr}}{r} + k^2\frac{e^{ikr}}{r}\hat{i}_x\right) - ikA_y^e\,\nabla\frac{e^{ikr}}{r}\wedge\hat{i}_y\right\}.$$

For the particular case of back scattering ($x = y = 0$) these expressions reduce to

$$\underline{E}^s\Big|_{x=y=0} = -E^i\left\{A_y^e\left(\frac{ik}{z} - \frac{1}{z^2} + k^2\right) - A_y^m\left(\frac{ik}{z} + k^2\right)\right\}\frac{e^{ikz}}{z}\hat{i}_y$$

$$\underline{H}^s\Big|_{x=y=0} = -\sqrt{\frac{\epsilon}{\mu}}\,E^i\left\{A_x^m\left(\frac{ik}{z} - \frac{1}{z^2} + k^2\right) - A_y^e\left(\frac{ik}{z} + k^2\right)\right\}\frac{e^{ikz}}{z}\hat{i}_x$$

and the back scattering cross section

$$\sigma = \lim_{z\to\infty} 4\pi z^2 \left| \frac{\underline{E}^s}{\underline{E}^i} \right|^2 = 4\pi k^4 \left(A_y^e - A_x^m\right)^2 . \tag{3.14}$$

If the spheroid is reduced to a sphere, i.e. if we let $F \to 0$ and $\xi_o \to \infty$ while $F\xi_o$ remains fixed, then

$$\xi_o^2 - \frac{\xi_o\left(\xi_o^2 - 1\right)}{2} \log \frac{\xi_o + 1}{\xi_o - 1} \to \frac{2}{3}$$

and
$$A_y^e \to -\frac{3V}{4\pi}, \quad A_x^m \to \frac{3V}{8\pi}$$

and the expression (3.14) for the cross section thus reduces to the well-known Rayleigh cross section of a sphere,

$$\sigma = \frac{4}{\pi} k^4 V^2 \left(\frac{9}{8}\right)^2 .$$

Before proceeding with the derivation of subsequent terms in the series, we note a simple argument given by Siegel (1959) which leads to an approximation to the Rayleigh coefficient obtainable with very little effort. This is based on the consideration that when the wavelength is much larger than the body dimensions, the details of form are not distinguishable and the principal effect of the body depends only on its size, i.e. volume. The dominant term in the scattered field should thus be expressible in terms of the volume plus a correction factor indicative of the general shape. This is verified in the following manner for the case of a plane wave incident on a perfectly conducting surface along the axis of symmetry.

The general multipole expansion of the scattered or radiated field shows that for the Rayleigh region a good approximation to the far-zone field is given by the dipole term alone. This in turn can be found by integrating the field strength multiplied by the moment axis over the surface, if the former is known. If the observation point is on the axis of symmetry, the integral reduces at once to the form

$$\int_0^{\ell} \pi \rho^2 \, a(z) \, dz$$

where ρ, z are cylindrical coordinates of the surface, ℓ is the length, and a(z) is the amplitude of the field on the surface. (The electric and magnetic fields are treated in identical fashion and contribute equally to the scattering cross section). If $\rho_{max} << 1$, i.e. if the body is elongated, then a(z) is slowly varying over the range of integration and may be approximated by a constant which, in analogy with the case of a plane surface, we may take to be twice the amplitude of the incident field. Under these assumptions the far-field amplitude of the electric vector, which is the sum of the contributions of the electric and magnetic dipoles, becomes

$$E = \frac{1}{\pi} k^2 E^i V$$

where E^i is the incident amplitude and V is the volume. For a general spheroid, prolate or oblate, the correction factor can be ascertained by comparison with the exact Rayleigh result. This is given by Siegel in the form

$$1+\frac{b}{\pi a}e^{-a/b}$$

where a is the axis of symmetry and b is the transverse axis. The agreement with the true Rayleigh coefficient is within one percent for any eccentricity. The nose-on backscattering cross section is then

$$\sigma = \frac{4}{\sigma}\ k^4\ V^2\left(1+\frac{b}{\pi a}e^{-a/b}\right)^2 .$$

It is not at once apparent how Rayleigh's formulation could be used to derive subsequent terms in the low frequency expansion. The problem becomes surprisingly involved as soon as the dynamic terms are introduced, and the details of the existing solutions are too voluminous to be included here in their entirety. We will limit ourselves to a general description of two independent extensions of Rayleigh's result, which more or less parallels the account given by Justice (1956).

The two methods to be described are those of Tai (1952) and Stevenson (1953) and following Justice we will refer to them as the vector mode function method and the potential function method respectively. Both solutions are based on the assumption of power series representations for incident and scattered fields of the form

$$\underline{E}^{i,s} = \sum_{n=o}^{\infty} \underline{E}_n^{i,s}(ik)^n$$

$$\underline{H}^{i,s} = \sum_{n=o}^{\infty} \underline{H}_n^{i,s}(ik)^n \qquad (3.15)$$

and the applicability of the results is naturally limited to cases where these representations are valid. Since the power series representation is unique

provided it exists, the two methods must produce equivalent results in the region where both are applicable. They differ, however, in generality and range of applicability. The potential function method is superior in these respects and can theoretically be applied to any body for which the requisite potential problems can be solved, with arbitrary incident field and material characteristics, and it can be carried out to any order desired. The solution is given in detail by Stevenson (1953b) for a general ellipsoid of arbitrary material with plane wave incident in any direction, carried out to the third order (the second order term vanishing as in the scalar case.) The vector mode function method becomes extremely complicated for off-axis incidence and is apparently not applicable for terms beyond the third. It was originally applied by Tai to a perfectly conducting oblate spheroid with symmetrical incidence, and subsequently to a prolate one with the same excitation by Justice. To facilitate the description and comparison of the methods, we will consider here only the latter problem. The more general results of Stevenson are tabulated in the appropriate section below.

If the material constants of the media are incorporated in the metrics of the field vectors (i.e. if Gaussian units are used), then Maxwell's equations can be written in the form

$$\nabla \wedge \underline{E} = ik\underline{H}, \quad \nabla \wedge \underline{H} = -ik\underline{E}$$

$$\nabla \cdot \underline{E} = \nabla \cdot \underline{H} = 0 \qquad (3.16)$$

Also if $\hat{n}$ is the unit normal to the scattering surface, then for a perfect conductor, the boundary conditions take the form

$$\hat{n} \wedge \underline{E}^s = -\hat{n} \wedge \underline{E}^i, \quad \hat{n} \cdot \underline{H}^i = -\hat{n} \cdot \underline{H}^s \tag{3.17}$$

Equations (3.16) and (3.17), along with the radiation condition on the scattered wave, constitute the mathematical statement of the problem, and on combining them with (3.15) and equating coefficients of like powers of k, there results the set

$$\left.\begin{aligned} \nabla \wedge \underline{E}_o^{i,s} = \nabla \wedge \underline{H}_o^{i,s} = \nabla \cdot \underline{E}_n^{i,s} = \nabla \cdot \underline{H}_n^{i,s} = 0 \\ \nabla \wedge \underline{E}_n^{i,s} = \underline{H}_{n-1}^{i,s}, \quad \nabla \wedge \underline{H}_n^{i,s} = -\underline{E}_{n-1}^{i,s} \end{aligned}\right\} \text{ everywhere} \tag{3.18}$$

$$\hat{n} \wedge \underline{E}_n^s = -\hat{n} \wedge \underline{E}_n^i, \quad \hat{n} \cdot \underline{H}_n^s = -\hat{n} \cdot \underline{H}_n^i \text{ on the surface.} \tag{3.19}$$

Furthermore, by the divergence theorem,

$$\int \hat{n} \cdot \underline{E}_n^{i,s} \, ds = \int \hat{n} \cdot \underline{H}_n^{i,s} \, ds = 0 \tag{3.20}$$

where the integration is over the surface of the scatterer.

These equations form the basis of both methods of solution, and despite their apparent simplicity, it develops that the procedures required and the forms evolved in either method rapidly become highly complex and voluminous for the higher order terms, so that we must limit ourselves here to a general description and refer the reader to the above-mentioned sources for the details of the methods.

In the vector mode function method, the next step is the repeated application of the curl operator to certain of the equations (3.18), to yield at once the vector differential equations

$$\begin{aligned} \nabla \wedge \nabla \wedge \underline{E}_1^s &= \nabla \wedge \nabla \wedge \underline{H}_1^s = 0 \\ \nabla \wedge \nabla \wedge \nabla \wedge \underline{E}_2^s &= \nabla \wedge \nabla \wedge \nabla \wedge \underline{H}_2^s = 0 \end{aligned} \tag{3.21}$$

The problem then is essentially that of representing the incident and scattered fields in terms of solutions of these equations and the first of (3.18) which have the proper types of radial dependence and which permit the satisfaction of the boundary conditions on the scatterer. Considering the limited available knowledge of general solutions of these types of equation, this process is necessarily more inductive than deductive, and it is easily inferred that a thorough study of the intimate characteristics of the spheroidal system must have been required for its completion. The process starts with the formation of two sets of spheroidal harmonics, $\phi_o^{i,s}$ and $\psi_o^{i,s}$ whose gradients satisfy the boundary conditions on $\underline{E}_o^{i,s}$ and $\underline{H}_o^{i,s}$ respectively. These gradients automatically satisfy the vector equations given above as well, and from them more vector solutions can be formed in a manner similar to the construction of Hansen's vector wave functions. The spheroidal harmonics, as pointed out in Section 2.1, are easily constructed from Legendre and trigonometric functions. Specifically, we can write

$$\phi_{mn} = P_n^m(\eta) \begin{Bmatrix} P_n^m(\xi) \\ Q_n^m(\xi) \end{Bmatrix} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \tag{3.22}$$

with the choice of P or Q functions determined by the desired behavior as $\xi \to \infty$, and that of cos or sin by the required ϕ-dependence. The gradients of the first two sets of these potential functions are required to satisfy the boundary conditions (3.19) on $E_o^{i,s}$ and $H_o^{i,s}$, and the additional vector solutions necessary for the representation of the higher order terms in the field expansions are expressed as linear combinations of certain of these harmonic functions and their gradients, multiplied in the appropriate manner by certain rectangular or spherical coordinate vectors. The choice of an adequate set of such functions for the representation of the incident and scattered fields and the proper construction of this representation is an inductive process too complicated to be described here in detail. In general it entails the expression of the first three terms in the incident field expansion in terms of five distinct vector mode functions, chosen on the basis of their angular dependence, and several arbitrary constants not uniquely determined by the incident field alone. Each of these five functions is associated with a corresponding function in the scattered field, and the boundary conditions, including the radiation condition on the scattered field, are applied to each mode individually. If the various functions and combinations have been properly chosen this process determines uniquely all the constants appearing, and the solution of the problem is complete out to the third

term in the power series expansion. It should be noted that aside from the tremendous increase in complication which would result from consideration of terms of still higher order, the method would apparently break down completely, since it has been shown by Stevenson that the terms beyond the third in the scattered field do not satisfy the radiation condition individually, but only collectively, and without this condition on each mode it is impossible, by the present method at least, to determine all the unknown constants involved. Some of the explicit forms evolved in this solution are tabulated in the appropriate section below.

In the potential function method developed by Stevenson the first steps are the same as in the previous method. It is a trivial matter to find potential functions ϕ_o^i, ψ_o^i whose gradients match the first terms of the incident field expansions, and the first equations of (3.18) and (3.19), together with the required behavior at infinity, then define standard Dirichlet and Neumann problems for the potentials ϕ_o^s, ψ_o^s, respectively, such that $\underline{E}_o^s = \nabla \phi_o^s$, $\underline{H}_o^s = \nabla \psi_o^s$. The next stage, however, cannot be reduced to potential problems alone, since $\underline{E}_1^s$, and $\underline{H}_1^s$ are not irrotational vectors. The procedure is to write each of these as the sum of an irrotational and a solenoidal component. The electric vector, for example, is written

$$\underline{E}_1^s = \underline{F}_1 + \nabla \phi_1^s \tag{3.23}$$

where $\underline{F}_1$ has zero divergence, vanishes at infinity, and satisfies the equation

$$\nabla \wedge \underline{F}_1 = \underline{H}_o^s \tag{3.24}$$

and ϕ_1^s is therefore an external harmonic function which must satisfy the boundary condition

$$\hat{n} \wedge \nabla \phi_1^s = -\hat{n} \left[\underline{F}_1 + \underline{E}_1^i \right] \tag{3.25}$$

Determination of ϕ_1^s is thus again a standard potential problem, once the particular solution $\underline{F}_1$ of (3.24) having required properties is found. If the right hand side of an equation of the form (3.24) vanishes at infinity at least to order r^{-3} and if its divergence is zero as well as the integral of its normal component over the scattering surface, then we can write an integral expression for the solution which, since $\underline{H}_o^s$ satisfies these conditions, in this case has the form

$$\underline{F}_1 = \nabla \wedge \frac{1}{4\pi} \int \underline{H}_o^s \cdot \frac{1}{r} \, dv \tag{3.26}$$

where dv is the volume element, r is the distance between observation and integration points and the integration covers the entire space, including the interior of the scatterer. In order to complete the definition of this integral, that of $\underline{H}_o^s$ must be extended to cover the interior of the body. Since $\underline{H}_o^s$ satisfies the latter equation of (3.20) this can be done by finding an internal harmonic function ψ_{oi}^s such that

$$\hat{n} \cdot \nabla \psi_{oi}^s = \hat{n} \cdot \underline{H}_o^s = \hat{n} \cdot \nabla \psi_o^s \tag{3.27}$$

on the surface, which is an ordinary Neumann potential problem. By making use of the form

$$\nabla \wedge \int \underline{H}_o^s \left(\frac{1}{r}\right) dv = \int \underline{H}_o^s \wedge \nabla \left(\frac{1}{r}\right) dv$$

we can write finally

$$\underline{E}_1^s = \frac{1}{4\pi}\left[\int_E \nabla \psi_o^s \wedge \nabla\left(\frac{1}{r}\right) dv + \int_I \nabla \psi_{oi}^s \wedge \nabla\left(\frac{1}{r}\right) dv\right] + \nabla \phi_1^s \qquad (3.28)$$

where the first integral covers the exterior of the scatterer and the second the interior, and ϕ_1^s is the potential satisfying (3.25) with $\underline{F}_1$ given by (3.26). The magnetic vector $\underline{H}_1^s$ is constructed in analogous fashion and can be written

$$\underline{H}_1^s = \frac{1}{4\pi}\left[\int_E \nabla \phi_o^s \wedge \nabla\left(\frac{1}{r}\right) dv + \int_I \nabla \phi_{oi}^s \wedge \nabla\left(\frac{1}{r}\right) dv\right] + \nabla \psi_1^s \qquad (3.29)$$

with ϕ_o^s, ϕ_{oi}^s, ψ_1^s defined by standard potential problems as before.

The procedure for finding the next term in each series is similar to the above, but here the situation is complicated by the fact that $\underline{H}_1^s$ and $\underline{E}_1^s$ vanish at infinity only to order r^{-2}, and the integrals corresponding to (3.26) are accordingly divergent. This difficulty can be overcome by constructing another pair of external harmonics ϕ_{1e}^s, ψ_{1e}^s whose normal derivatives match those of $\underline{E}_1^s$ and $\underline{H}_1^s$ on some surrounding surface ξ_1 exterior to the scatterer, which can be arbitrarily large. We can then write for the electric field

$$\underline{E}_2^s = \underline{F}_2 + \nabla \phi_2^s \qquad (3.30)$$

with $\underline{F}_2$ given in the region $\xi_o \leq \xi \leq \xi_1$ by the form

$$\underline{F}_2 = \int_{\xi_o \leq \xi < \xi_1} \underline{H}_1^s \wedge \nabla\left(\frac{1}{r}\right) dv + \int_{\xi < \xi_o} \nabla\, \psi_{1i}^s \wedge \nabla\left(\frac{1}{r}\right) dv + \int_{\xi > \xi_1} \nabla\, \psi_{1e}^s \wedge \nabla\left(\frac{1}{r}\right) dv \quad . \tag{3.31}$$

(Here ψ_{1e}^s is an internal harmonic whose normal derivative matches that of ψ_1^s on ξ_o.) For $\xi > \xi_1$, however, this expression is of no use, since here it yields $\nabla \wedge \underline{F}_2 = \nabla\psi_{1e}^s \neq \underline{H}_1^s$, and thus the function ϕ_2^s is not yet determinable as an external harmonic. We are forced to resort to using another type of expression for the field vectors consisting of surface integrals, of the form

$$\underline{E}^{i,s} = ik\int \hat{n} \wedge \underline{H}^{i,s}\phi \; ds + \nabla\wedge \int \hat{n} \wedge \underline{E}^{i,s}\phi \; ds \; - \nabla \int \hat{n} \cdot \underline{E}^{i,s}\phi \; ds \tag{3.32}$$

where $\phi = \frac{e^{ikr}}{4\pi r}$, r is the distance from a point on the surface to the field point, and the integration covers the scattering surface in each term. When the field expressions (3.15) and the standard exponential series are substituted here and coefficients of like powers of k are collected, there results a set of equations in the components $\underline{E}_n^{i,s}$, $\underline{H}_n^{i,s}$, of which the pertinent one for $\underline{E}_2^s$ is

$$4\pi\, \underline{E}_2^s = \int \frac{1}{r}\, \hat{\xi} \wedge \underline{H}_1^s \; ds + \nabla \wedge \int \frac{1}{r}\, \hat{\xi} \wedge \underline{E}_2^s \; ds + \frac{1}{2}\nabla \wedge \int r\, \hat{\xi} \wedge \underline{E}_o^s \; ds$$
$$- \nabla \int \frac{1}{r}\, \hat{\xi} \cdot \underline{E}_2^s \; ds - \frac{1}{2}\nabla \int r\, \hat{\xi} \cdot \underline{E}_o^s \; ds \tag{3.33}$$

(The remaining terms which apparently enter can easily be shown to vanish.) The second term can be written in terms of the known functions $\underline{E}_2^1$ by virtue of the

boundary condition (3.19), and substituting (3.30) in the fourth term, we can write

$$\underline{E}_2^s = \underline{F}'_2 + \nabla \phi_2^{s'} \tag{3.34}$$

with

$$4\pi \underline{F}'_2 = \int \frac{1}{r} \hat{\xi} \wedge \underline{H}_1^s \, ds - \nabla \wedge \int \frac{1}{r} \hat{\xi} \wedge \underline{E}_2^i \, ds$$
$$+ \frac{1}{2} \nabla \wedge \int r \hat{\xi} \wedge \underline{E}_o^s \, ds - \frac{1}{2} \nabla \int r \hat{\xi} \cdot \underline{E}_o^s \, ds$$
$$- \nabla \int \frac{1}{r} \hat{\xi} \cdot \underline{F}_2 \, ds \tag{3.35}$$

and

$$\nabla \phi_2^{s'} = - \nabla \int \frac{1}{r} \hat{\xi} \cdot \nabla \phi_2^s \, ds. \tag{3.36}$$

The value (3.31) for $\underline{F}_2$ can now be used in the last term of (3.35) and $\underline{F}'_2$ is thus completely determined, and furthermore it is easily shown that both $\underline{E}_2^s$ and $\underline{F}'_2$ vanish at infinity, so that $\phi_2^{s'}$ is an external harmonic function and is thus determinable by means of the boundary condition given by (3.19), (3.34), and (3.35). The determination of $\underline{E}_2^s$ is now complete, and that of $\underline{H}_2^s$ is perfectly analogous.

Stevenson makes the statement that the general method described here can be carried out to any order desired. However, the success of the method with higher order terms depends on the ability to find particular functions $\underline{F}_n$ to represent the solenoidal components of each term, and furthermore the fact that the field components $\underline{E}_n^s$, $\underline{H}_n^s$ do not vanish at infinity for $n > 2$ renders the problem of finding

the irrotational components $\nabla \phi_n^s$ more difficult. The details of how these difficulties might be overcome have not been published.

Another difficulty with this representation arises when the far field is considered. The behavior of the higher order terms indicates that the given series becomes useless as r increases without limit. A new representation, however, which is valid everywhere outside a large sphere surrounding the scatterer, can be derived in a manner similar to that used by Rayleigh to obtain the far field representation from that of the near field. In the paper of Stevenson that is accomplished by writing the general expressions for the components of an exterior E (or TM) wave and an H(or TE) wave in terms of spherical wave functions and expanding the radial components in double power series in k and R. Each coefficient in the radial component of the near-field series determined earlier is then expanded in powers of R and the two expansions thus obtained are compared, term by term, yielding a general relation between the individual surface harmonics involved in the expression of the far field and those of the near field. Once the latter are obtained from the previous analysis, the far-field expressions are easily written down. It develops also that no accuracy is lost in passing from the near to the far field, i.e., knowledge of N terms in the near-field series gives at once N terms of the far-field series. The explicit expressions for the far field are given in Section 4.1.

Still another method of deriving the vector Rayleigh series is described by Senior (1964). This is perhaps more straightforward and schematically simple

than either of the above methods, and there are no analytical difficulties in carrying it out to any arbitrary degree; however, again the quantity of labor involved rapidly approaches a prohibitive level as the number of terms increases, and to date the forms have been worked out only as far as those in the previous solutions, and only for a conducting spheroid with plane wave incident nose-on.

The first step in this procedure is to expand the incident and scattered fields in terms of appropriate sets of Hansen's vector wave functions. The question of optimum choice of these sets is still more or less open, but for reasons of simplicity and generality the ones chosen were based on the radius vector $\underline{r}$, and with this basis and with the assumed incident field, the only sets of vector functions resulting are the $\underline{M}\underset{o}{e}_{1n}$ and $\underline{N}\underset{o}{e}_{1n}$, as defined in (2.43) with $\underline{r}$ replacing $\underline{a}$. In particular the scattered electric field has the representation

$$\underline{E}^{s} = \sum_{n=1}^{\infty} \left(A_n \underline{M}o_{1n} + B_n \underline{N}e_{1n} \right) \tag{3.37}$$

with the coefficients A_n, B_n as yet undetermined. The vector functions may be expressed in terms of the prolate spheroidal coordinates by formulas analogous to those of (2.49), and using these forms and the explicit expression for the incident electric field, which is

$$\underline{E}^{i} = -\hat{i}_{\eta}\, \eta \sqrt{\frac{\xi^2 - 1}{\xi^2 - \eta^2}}\; e^{-ic\xi\eta} \cos\phi - \hat{i}_{\phi}\, e^{-ic\xi\eta} \sin\phi, \tag{3.38}$$

the boundary conditions on the surface (2.46) become, after some manipulation,

$$\sum_{n=1}^{\infty} A_n \Big\{ \xi \left[(\lambda_{1n}-c^2\eta^2)(\xi^2-1)+1\right] (\xi^2-\eta^2) S_{1n}(c,\eta) R_{1n}^{(3)'}(c,\xi)$$

$$-2\xi\eta(\xi^2-1)(1-\eta^2) S'_{1n}(c,\eta) R_{1n}^{(3)}(c,\xi) - 2(\xi^2-1)^2\eta^2 S_{1n}(c,\eta) R_{1n}^{(3)'}(c,\xi)$$

$$+(\xi^2-1)(\xi^2-\eta^2)(1-\eta^2)\left[S_{1n}(c,\eta)+\eta S'_{1n}(c,\eta)\right] R_{1n}^{(3)'}(c,\xi)\Big\}$$

$$=\sum_{n=1}^{\infty} B_n\, c\,\eta(\xi^2-\eta^2)\, S_{1n}(c,\eta)\, R_{1n}^{(3)}(c,\xi) - (-i)^{n-1}\frac{(n-1)(c\eta\xi)^{n-1}}{\eta(n-1)!}(\xi^2-\eta^2)\sqrt{(\xi^2-1)(1-\eta^2)}$$

and $$(3.39)$$

$$\sum_{n=1}^{\infty}\Big\{\frac{B_n}{c}\left[S_{1n}(c,\eta)R_{1n}^{(3)}(c,\xi)+\frac{\xi(\xi^2-1)}{\xi^2-\eta^2}S_{1n}(c,\eta)R_{1n}^{(3)'}(c,\xi)+\frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}S'_{1n}(c,\eta)R_{1n}^{(3)}(c,\xi)\right]$$

$$-A_n\frac{(\xi^2-1)(1-\eta^2)}{\xi^2-\eta^2}\left[\xi S'_{1n}(c,\eta)R_{1n}^{(3)}(c,\xi)-\eta S_{1n}(c,\eta)R_{1n}^{(3)'}(c,\xi)\right]$$

$$+(-i)^{n-1}\frac{(c\xi\eta)^{n-1}}{(\eta-1)!}\sqrt{(\xi^2-1)(1-\eta^2)}\Big\}=0$$

where ξ is the coordinate of the scattering surface and the primes indicate differentiation with respect to ξ or η. All quantities appearing here which depend on c are now expanded in power series. These include the coefficients A_n and B_n, the eigenvalue λ_{1n}, and the spheroidal functions, which must be expressed in terms of the corresponding sets of spherical ones. The magnitude of the task now becomes apparent. It is not hard to show, however, that once these expansions are inserted and the coefficients of like powers of c collected, the result is an

essentially triangular system of equations in the coefficients in the power series expansions of A_n and B_n, so that these can be determined sequentially out to any order desired. Again, the explicit forms will be tabulated later.

One more attack on the scalar problem might be discussed briefly here, though from the standpoint of generality and rate of convergence it might equally well be classed as an exact solution. This is the Schwinger variational technique* for the solution of an integral equation, as applied to the problem of a hard spheroid with nose-on plane wave incidence by Sleator (1960). The formulation of the integral equation for the velocity potential of the total field is standard procedure, and the equation may be written

$$\phi(r) = e^{-ikz} - \frac{1}{4\pi} \int_S \phi(r') \frac{\partial G(r, r')}{\partial n'} \, da' \tag{3.40}$$

where r is the field point, r' the source point, $\frac{\partial G(r, r')}{\partial n'}$ is the normal derivative of the free space Green's function, and the integral covers the scattering surface S. Direct application of the boundary condition

$$\frac{\partial}{\partial n} \phi(r) \Big|_S = 0 \tag{3.41}$$

to (3.40) gives

$$\frac{\partial}{\partial n} e^{-ikz} \Big|_S = \frac{1}{4\pi} \int_S \phi(r') \frac{\partial^2}{\partial n \partial n'} G(r, r') da' \tag{3.42}$$

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*It has been shown by Jones (1956) that this technique is in general equivalent to a method developed earlier by Galerkin (1915) for the solution of certain types of integral equations.

(The formal differentiation under the integral is not obviously legitimate, since the resulting integral is apparently divergent. This difficulty, however, can be overcome by proper treatment of the ensuing forms, and a slightly more complicated but equivalent formulation would obviate it completely.) The Schwinger technique is to define next the quantity

$$J\left[\phi\right] \equiv \frac{\iint_S \phi(r)\frac{\partial^2 G}{\partial n \partial n'}(r, r')\phi(r')\,da\,da'}{\left[\int_S \phi(r)\frac{\partial}{\partial n} e^{-ikz}\,da\right]^2}, \tag{3.43}$$

and it can be shown that the potential $\phi(r)$ is then the solution of the variational problem $\delta J\left[\phi\right] = 0$. It also follows that the total backscattering cross section is given by the formula

$$\sigma = 4\pi \left| J_o \right|^{-2} \tag{3.44}$$

where J_o is the stationary value of $J\left[\phi\right]$. It might be noted here that the usual procedure with this mechanism is to assume a simple trial function for the surface potential $\phi(r')$, for which the integrals are more or less manageable. Since the error in the result is proportional to the square of that in the trial function, the calculated function should be more accurate than the original one, and the process can be iterated if necessary. In certain cases of separable geometries, however, the iterative scheme can be replaced by an expansion process. Specifically, if the unknown function $\phi(r)$ is expanded in terms of an appropriate set of angle functions

with coefficients which, in a symmetrical problem, are functions only of the radial variable, application of the stationary condition to $J[\phi]$ yields at once an infinite system of equations for these coefficients, whose solution, if it exists, gives an exact representation of the unknown function $\phi(r)$. As a matter of fact, if the basis functions used are the standard orthogonal eigenfunctions of the problem in question and if the Green's function is similarly expanded, then the integrals are all tractable, the infinite system is diagonal, and the solution immediately reduces to the standard wave function solution discussed previously.

In the work of Sleator, however, the spheroidal functions, which are the natural basis for the expansions, are by-passed in an effort to simplify the numerical treatment, and the potential is expanded directly in Legendre functions,

$$\phi(\xi, \eta) = \sum A_{\mu}(\xi)\, P_{\mu}(\eta) \; . \tag{3.45}$$

Substitution into (3.43) and application of the stationary condition

$$\frac{\partial J}{\partial A_{\mu}} = 0 \text{ for all } \mu$$

yields the infinite system

$$\sum_{\mu=0}^{\infty} A_{\mu}\, C_{\mu\nu} = 4\pi B_{\nu} \qquad \nu = 0,\, 1,\, 2 --- \tag{3.46}$$

where

$$C_{\mu\nu} \equiv \int_S P_{\mu}(\eta) \frac{\partial^2}{\partial n\, \partial n'} G(r, r')\, P_{\nu}(\eta')\, da\, da' \tag{3.47}$$

$$B_\nu \equiv \int_S P_\nu(\eta) \frac{\partial}{\partial n} e^{-ikz} \, da . \tag{3.48}$$

If a Fourier integral representation is used for the Green's function in (3.47), then by rearranging the 7-fold integral that results, it is possible to carry out all but one of the integrations in closed form. The last integration, however, is apparently best handled by numerical or graphical techniques. The system (3.46) can be proved convergent and can therefore be solved in truncated form to any order desired. The integral in (3.48) is immediately obtainable from known forms.

It is thus possible to obtain an exact solution to the spheroidal scattering problem without resorting to the spheroidal wave function, but the amount of labor involved in evaluating the integrals (3.47) and solving the system (3.46) make it questionable whether this method is preferable to the one previously described. At any rate, the number of terms required in the series (3.45) increases with the frequency in the same manner as in the wave-function solution, and the quantity of labor involved rises much more rapidly, so that for practical purposes the variational solution is perhaps justly classified as a low-frequency approximation.

An analogous formulation of the vector problem is much more complicated and leads to integrals which appear prohibitively difficult to evaluate.

3.1.2 High Frequency Approximations

As indicated in the previous section, the extension of low-frequency approximations and techniques to cover the regions where the body dimensions exceed or

even approach the wavelength of the incident radiation is fraught with difficulties of several kinds, principal of which are the inordinate amount of labor required in deriving the successive terms in the field expansions and the limited range of convergence of the results. One might hope for better luck at the other end of the spectrum, and indeed the situation does turn out to be more favorable there. Various methods based on optical laws have been developed in considerable generality, and when applied to the spheroid problem some of these produce reasonably good approximations which, under certain circumstances at least, extend well into the resonance region. These circumstances usually involve limitations on some other parameter, however, so that it cannot be said that the problem is completely solved. Before going into these combined restrictions, we will mention briefly the limiting forms of the exact solution when the frequency increases indefinitely. In the interest of simplicity, we consider first only bodies which are perfect conductors. The modifications of the theory required to cover dielectric bodies will be developed later.

3.1.2.1 Geometric and Physical Optics

The ultimate form of any scattering phenomenon as the wavelength decreases (which form can of course be termed the first approximation for small but finite wavelength) is completely describable in terms of the laws of geometric optics. In this limit the scattered field of any smooth convex conducting body is determined

at any exterior point entirely by the curvature of the body at the specular point, i.e. that point on the surface where an incident ray and a reflected ray through the observation point are coplanar with the surface normal and make equal angles with it. It is not hard to show that if R_1 and R_2 are the principal radii of curvature at this point, then the scattering cross section $\sigma_{g.o.}$ is given by the expression

$$\sigma_{g.o.} = \pi R_1 R_2 .$$

At the tip of a prolate spheroid, the principal radii are of course equal and have the value b^2/a, so that for nose-on backscattering we can write

$$\sigma_{g.o.} = \pi b^4 / a^2$$

and this is customarily used as a normalization factor for values of σ obtained otherwise. These results are also derivable in terms of a limit for vanishing wavelength of a more general, frequency dependent result (see, for example, Siegel et al 1955), which is considered below. Also it is shown by Crispin et al (1959) that in this limit for sufficiently smooth bodies, the geometric optics cross section with transmitter and receiver separated by an angle β is equal to that observed if both are located on the bisector of this angle. In the former reference an expression is derived for the geometric optics cross section as a function of the separation angle β with transmitter located in the axis of symmetry. For the range $\beta < \pi$ this is

$$\sigma(\beta) = 4\pi b^4 a^2 \left[a^2(1+\cos\beta) + b^2(1-\cos\beta)\right]^{-2} . \qquad (3.49)$$

By the above theorem, this expression also gives the monostatic cross section if transmitter and receiver are both located at an angle $\beta/2$ with respect to the major axis of the spheroid.

The question of the accuracy of these results and the lower limit of the frequency range in which they can reasonably be applied is not easily answered. Undoubtedly this depends on the eccentricity of the body and also on the directions of incidence and observation. Some indication of this is furnished by the fact that the geometric optics result for nose-on backscattering from a paraboloid, which is one limiting form of a spheroid as the eccentricity approaches unity, is indeed exact. The scarcity of data, either theoretical or experimental, at high frequencies makes it difficult to establish in general where the optical laws become dominant. Recent work on the scalar problem for a spheroid of axis ratio 10:1 (Goodrich and Kazarinoff, 1962) indicates that there are resonance phenomena occurring even in the range $ka \simeq 100$ for this body, and it is clear that in general the geometric optics result is of limited value in most practical problems. The underlying principles, however, form the basis of a more refined geometric approach which will be discussed presently and which has proved both physically illuminating and practically useful.

Before dealing with the latter, we will consider a somewhat simpler but still sometimes useful approach based on Huygen's principle (otherwise known as Kirchhoff theory.) This is the well-known physical optics solution, which is

discussed at length for the sphere problem in the first report of this series. The qualitative aspects of the spheroid solution are generally similar to those of the sphere though the specific forms are of course more complicated and, to the best of our knowledge, have not been worked out in as great detail. Accordingly we here content ourselves with a rather brief formulation and listing of the available results.

The essentials of the Kirchhoff theory can be embodied in the formula

$$\underline{H}^{S} = \frac{1}{4\pi} \int_{S} (\hat{n} \wedge \underline{H}) \wedge \nabla \left(\frac{e^{ikR}}{R} \right) ds \qquad (3.50)$$

where $\underline{H}$ is the total magnetic field on the surface, (here the tangential component may be used since the normal component is eliminated by the vector product,) $\hat{n}$ is the unit normal out of S, and R is, as usual, the distance from a point on the surface to the field point. The integration covers the surface, and if the true value of $\underline{H}$ is employed, the expression is exact. The physical optics approximation, however, which represents the principal utility of the form, is based on the substitution of an approximate value of $\underline{H}$, specifically the value given by the geometric theory for a locally plane surface, which is twice the tangential component of the incident magnetic field in the illuminated region and zero in the shadow region. Even with this approximation, the evaluation of the integration (3.50) is not trivial in general, since except in the case of symmetrical incidence, the shadow curve, which bounds the region of integration, involves both angular coordinates, and the

quantity R is itself a complicated expression. If we consider only the far-field scattering cross section, however, some simplification is possible, and if the incident field is required to be a plane wave propagating along the axis of symmetry, a simple result is easily obtained for the backscattering cross section.

Consider first the case where the transmitter is located in the axis of symmetry, which we take to be the z-axis, emitting a plane wave of unit amplitude with magnetic vector

$$\underline{H}^i = \hat{i}_x e^{-ik\hat{i}_z \cdot \underline{r}} \tag{3.51}$$

and the receiver is at a large distance from the scatterer, separated from the z-axis by an angle θ. If $\underline{r} = r\hat{r}$ is the position vector of the observation point and $\underline{r}'$ that of the integration point, then the gradient in (3.50) can be approximated by the form

$$\nabla\left(\frac{e^{ikR}}{R}\right) \cong \frac{e^{ikr}}{r}(-ik\hat{r})e^{-ik\hat{r}\cdot\underline{r}'}$$

and using this and the approximation specified above for the field $\underline{H}$ on the surface, we can write the scattered field, after some rearrangement, as

$$\underline{H}^S \cong \frac{ik}{2\pi}\frac{e^{ikr}}{r}\left[-\hat{i}_x(\hat{r}\cdot\underline{f}) + \underline{f}(\hat{i}_x\cdot\hat{r})\right] \tag{3.52}$$

with

$$\underline{f} = \int_{S'} \hat{n}\, e^{-ik\underline{r}'\cdot(\hat{i}_z+\hat{r})}\, ds' \tag{3.53}$$

S' being the illuminated region of the surface. For the case of backscattering, i.e. $\hat{r} = \hat{i}_z$, the second term in the bracket in (3.52) disappears and the far field amplitude, which is the coefficient of $-i_x \frac{e^{ikr}}{r}$ in (3.52), can be written

$$F(0) = \frac{i}{\lambda} \int_{S'} n_z e^{-2ikz'} ds \qquad (3.54)$$

where n_z is the z component of the outward normal and z' is the z coordinate of the integration point.

When the above formulas are applied to the prolate spheroid, the resulting expression for the nose-on backscattering cross section is easily found to be

$$\sigma = \frac{\pi b^4}{a^2} \left[1 - \frac{\sin 2ka}{ka} + \left(\frac{\sin ka}{ka} \right)^2 \right]. \qquad (3.55)$$

The function in brackets is plotted for a 10:1 spheroid, over a limited range of ka in Fig. 24, along with various other solutions. The expected discrepancies in the regions of large wavelength are apparent at once. For larger ka, the oscillation about the geometric optics value seems reasonable, but a close comparison of the analogous form for the sphere with the exact (wave-function) solution (see Crispin et al, 1959) indicates that there is little correlation in either phase or amplitude, at least until the oscillations in both solutions become very small. Also the above-noted results of Goodrich and Kazarinoff on resonance phenomena indicate that this may occur only at extremely large values of ka, at least for thin spheroids. It is

thus difficult to say in general what or where are the advantages of the physical optics result over that of geometrical optics.

For more general angles of incidence and observation the integral in (3.50) is not so tractable and few results are available. Application of the stationary phase principle yields only the geometric optics form (3.49) (see Siegel et al, 1955). For any given direction of incidence, however, there is one observation direction in which the integral can be evaluated exactly. This is the direction for which the normal to the plane of the shadow curve bisects the angle between transmitter and receiver, this occurs when

$$\tan\left(\frac{\beta}{2}\right) = \frac{a^2 - b^2}{\rho^2} \sin\beta_1 \cos\beta_1$$

where β is the angular separation between transmitter and receiver, β_1 is the angle between the axis of symmetry of the spheroid and the plane of the shadow curve, and

$$\rho^2 = a^2 \cos^2\beta_1 + b^2 \sin^2\beta_1$$

Letting $M \equiv k\rho\cos\left(\frac{\beta}{2}\right)$, one obtains the cross section in the form

$$\sigma = \frac{\pi a^2 b^4}{\rho^4}\left[1 - \frac{\sin 2M}{M} + \left(\frac{\sin M}{M}\right)^2\right]. \tag{3.56}$$

Here again it is difficult to judge the accuracy of this form in general on the basis of any available information.

There are two principal sources of error in the general physical optics procedure, one being the approximate evaluation of the integrals and the other the discrepancy between the assumed values of the field on the body and the true values. The former is essentially a computational problem, with which we will not concern ourselves at present. Rather we will consider certain modifications or refinements of the assumptions on the surface fields and the resulting corrections to the geometric or physical optics scattering coefficients. One such refinement is due to Jones (1957). In this article only the total scattering coefficient (total energy flux in the scattered wave divided by the energy flux in the incident wave striking the obstacle) is considered, and it is observed that in this regard, and in the optics region, the different regions of the surface contribute independently. The main weakness of the physical optics assumption on the surface current is in the region of the penumbra, i.e. the neighborhood of the shadow curve, where it is assumed to be discontinuous, in violation of the actual boundary condition. Jones accordingly assumes a different distribution in the penumbra region and determines its effect on the total scattering coefficient. For a smooth convex body the field in the penumbra is taken to be locally that of a cylinder whose generator is tangent to the shadow curve and whose radius of curvature is that of the given body in a plane normal to this tangent. The total contribution of the penumbra region to the scattering coefficient is then formed by integrating along the entire shadow curve.

Consider first the scalar problem and assume a plane incident wave with unit energy per unit area normal to its propagation direction. Then the total scattering coefficient arising from the illuminated region proper has the value 2, and if a cylinder of radius R is oriented so that its axis makes an angle $\frac{\pi}{2} - \beta$ with the incident direction, it can be shown by means of the exact solution that the energy scattered per unit length by the penumbra region is

$$b_o \left(\frac{R \cos \beta}{k^2} \right)^{1/3}$$

where b_o is a coefficient which incorporates the effect of the boundary condition, and whose values for the usual cases are given in the table of results hereafter. Applying this local analysis to a three-dimensional (convex) body, with the stipulation that the quantity, $kR \cos \beta$ must always be large, it follows that if D is the shadow curve, with differential arc length ds, and S_o the projected area of the body on a plane normal to the incident direction, then the total scattering coefficient is given by the formula

$$\sigma_T \cong 2 + \frac{b_o}{k^{2/3} S_o} \int_D R^{1/3} \cos^{1/3} \beta \, ds \tag{3.57}$$

which, for a prolate spheroid with nose-on incidence, reduces immediately to

$$\sigma_T \cong 2 + 2\, b \left(\frac{a}{kb^2} \right)^{2/3} . \tag{3.58}$$

For broadside incidence the integral is slightly more complicated, due to the variation of R, and yields a hypergeometric function, so that

$$\sigma_T \cong 2 + \frac{2b_o}{(kb)^{2/3}} \; {}_2F_1\left(-\frac{2}{3}, \frac{1}{2}; 1; 1 - \frac{b^2}{a^2}\right) . \qquad (3.59)$$

The treatment of the electromagnetic problem for a conducting body is somewhat more complicated. The contribution of the penumbra region of a cylinder must first be ascertained, making use of the proposition that if the incident plane wave is independent of the axial coordinate, then the total field can be decomposed into two parts, for one of which the electric vector is parallel to the axis and satisfies a Dirichlet boundary condition, and for the other the magnetic vector is in this direction and satisfies a Neumann condition. These components can accordingly be derived from the solutions of the standard scalar problems, as indicated in Kleinman and Senior (1963). In the present case if we write the scattering coefficient for the scalar Dirichlet problem with incident direction normal to the cylinder axis as

$$\sigma_D \cong 2 + b_D (kR)^{-2/3}$$

and that for the Neumann problem as

$$\sigma_N \cong 2 + b_N (kR)^{-2/3}$$

(i.e. let b_D and b_N be the specific values of the coefficient b_o referred to above) then it develops that the contribution of the penumbra region on one side in the

electromagnetic case, with incident direction making an angle $\frac{\pi}{2} - \beta$ with the cylinder axis and electric vector an angle γ, is

$$\left(\frac{R\cos\beta}{k^2}\right)^{1/3}\left[b_N + \left(b_D - b_N\right)\sec^2\beta\cos^2\gamma\right],$$

and accordingly the total electromagnetic scattering coefficient for a three-dimensional object, with the angles now referred to the tangent to the shadow curve in place of the cylinder axis, becomes

$$\sigma_T \simeq 2 + \frac{1}{k^{2/3} S_o}\int_D \left[b_N + \left(b_D - b_N\right)\sec^2\beta\cos^2\gamma\right](R\cos\beta)^{1/3}\,ds. \qquad (3.60)$$

For any solid of revolution with symmetric incidence and radius b of the shadow boundary this reduces at once to

$$\sigma_T \simeq 2 + \frac{R^{1/3}\left(b_D + b_N\right)}{k^{2/3}\, b} \qquad (3.61)$$

which is the average of the coefficients for the two scalar problems, and for the prolate spheroid, since $R = a^2/b$, it becomes

$$T \simeq 2 + \left(b_D + b_N\right)\left(\frac{a}{kb^2}\right)^{2/3}. \qquad (3.62)$$

The two coefficients for broadside incidence (i.e. electric vector parallel or perpendicular to axis of symmetry) are expressible in terms of hypergeometric functions, as in the scalar case, and the explicit forms are tabulated later.

3.1.2.2 Modified Geometrical Theory

The simplicity of the above development is of course due in a large part to the fact that it is concerned only with the total scattering coefficient. The problem of refining the optical techniques to give significant improvements in the differential scattering or radiation pattern results is considerably more complicated. Perhaps the most notable contributions in this direction are the theories developed by Fock (1946) (see also Goodrich, 1959) and Keller (cf. Levy and Keller, 1959). Both of these become rather involved for three-dimensional problems and depend more on physical arguments than on mathematical techniques. Both lead directly to the so-called creeping wave theory, which is also supported by the more mathematical derivation based on asymptotic expansions and the Watson transform to be discussed later, and all of these, at least in regions where they are applicable, produce essentially identical results, certain of which are presented in the appropriate section below. It is beyond the scope of the present effort to give the detailed derivations of these results, but we present here a brief account of the principal arguments and assumptions on which they are based. We will concern ourselves primarily with Keller's formulation since this has been worked out more explicitly and comprehensively than the others and thus appears to have a wider range of applicability. The theory has been developed in general terms for both vector and scalar problems involving smooth convex bodies of more or less arbitrary shape and material properties, and the particular results for the scalar spheroid problem with symmetrical

excitation and either Dirichlet or Neumann boundary condition have been given by Levy and Keller (1959), who also give the nose-on backscattered electric field, which in the optical limit is easily obtainable in terms of the scalar results. In addition, the case of a dielectric spheroid has been treated with similar methods by Thomas (1962).

The principal restriction in the theory in question, aside from the requirement of sufficiently small wavelength, is that the media involved should be individually homogeneous and isotropic, so that the radiant energy travels in straight lines normal to the wave front, except on the boundaries of the media, where it follows the geodesics in accordance with Fermat's principle. At each point of such a trajectory, or ray, the field has a well defined (vector or scalar) amplitude and phase. The latter is assumed to vary continuously and uniformly with the distance along the ray except at a focal point, where it suffers a drop of $\pi/2$. The amplitude is determined by the source of the ray and by the energy co ervation law as applied to the various phenomena which it may encounter. For a vector field, the direction of the amplitude must be normal to the ray, and it is assumed to remain constant except at a boundary, where it is governed by the usual laws of reflection and transmission. At any point in space, the total field is the sum of the fields on all rays passing through the point. These can be classified in one of four categories according to what befalls them between source point and field point: incident, if no interruption occurs; reflected, if an optical reflection occurs; refracted, if the ray

passes through more than one distinct medium; diffracted, if it follows a boundary for a finite distance. For a convex scatterer there is no overlapping of these categories unless the body is penetrable, in which case a refracted ray may also be reflected internally. The laws governing the behavior of the first three kinds of rays are familiar enough, but the fourth requires further comment. A diffracted ray is produced wherever an incident ray is tangent to a boundary surface. From such a point the ray follows a geodesic, at each point of which it splits and originates a new ray which leaves the surface tangentially at that point. Thus a diffracted ray from the source to a given field point consists in general of two straight line segments tangent to the obstacle plus a geodesic arc connecting the points of tangency and tangent to both lines.

The number of rays connecting a simple source with a given field point is in general finite and for simple configurations quite small, but there are exceptional regions, lines or surfaces, called caustics, which are envelopes or accumulation regions of families of rays from the source (they may alternatively be defined as the loci of centers of curvature of the wave fronts). For field points in the neighborhood of one of these, the sum referred to above apparently becomes infinite, and the theory must be modified in a manner to be noted below. The diffracting surface is itself a caustic, and in rotationally symmetric problems, the axis of symmetry is also one. For reflected rays the caustics are more complicated.

On the basis of the above, expressions are derived fairly easily for the field at a given point in terms of that at some preceding point on a ray connecting it to the source. In consideration of the fact that the energy flux through every cross section of a tube of rays is constant, it develops that if the amplitude and phase of the field at a point P_o are A_o, ϕ_o, then the field at the point P, a distance s further along the ray can be written

$$u(P) = A_o \left[\frac{\rho_1 \rho_2}{(\rho_1 + s)(\rho_2 + s)} \right]^{\frac{1}{2}} e^{ik(\phi_o + s)} \tag{3.63}$$

where ρ_1, ρ_2 are the principal radii of curvature of the wave front at P_o. (As noted above, if P_o and P lie on opposite sides of a caustic, there is an additional factor of $e^{-i\frac{\pi}{2}}$.) If P lies on a reflected ray, the point of reflection is taken as the reference point P_o and it is assumed that the field there is proportional to the incident field, the proportionality factor being the reflection coefficient, which is determined by the surface characteristics at the point. (If the field u is a vector field, then A is a vector and the reflection coefficient is a matrix.) At any field point P, then, the incident and reflected fields will have the general form (3.63), and the sum of these is referred to as the geometric field u_g.

The determination of the diffracted field is somewhat more difficult. The reference point for a surface ray is the point of tangency of the incident ray which

generates it, and here we assume that the field is proportional to the incident field, i.e. since the phase varies continuously the diffracted amplitude is written

$$A_d(P_o) = D(P_o)A_i(P_o) \tag{3.64}$$

with A_i the incident amplitude and D the diffraction coefficient, which is yet to be determined. Also, in accordance with the above assumptions, at each point on a surface ray, energy is being radiated into space at a rate which is assumed proportional to the square of the amplitude at the point times the elementary area, with proportionality factor α. This yields a differential equation in the amplitude as a function of distance s along the surface ray, whose solution is found immediately to be

$$A_d(s_1) = A_d(0)\sqrt{\frac{d\sigma_o}{d\sigma}}\ \exp\left[-\int_0^{s_1} \alpha(s)\,ds\right] . \tag{3.65}$$

Here $d\sigma_o$ is the width of an elementary strip containing the ray at the initial point $s = 0$ and $d\sigma$ its width at s_1, and the derivative notation signifies the limit of the ratio as the quantities approach zero. The decay coefficient $\alpha(s)$ must also be determined independently. The form (3.65) can be combined with (3.63) and (3.64) to give the field at any point on the surface ray in terms of that at a point Q on the incident ray (see Fig. 2) and the result can be applied to the point P_1 where a tangential ray through the field point P leaves the surface.

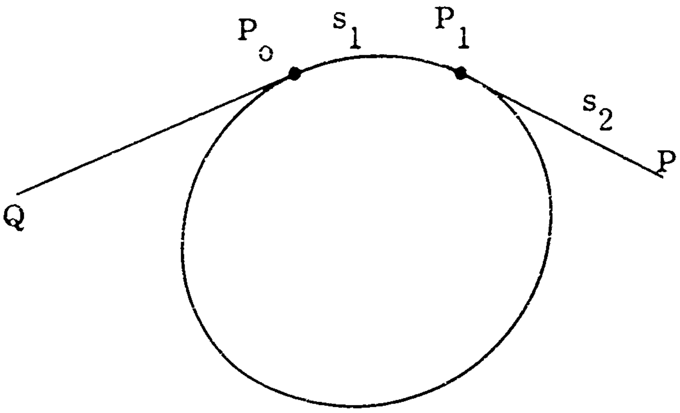

FIG. 2

A difficulty arises in applying (3.63) to find the field at P in terms of that of P_1, since the latter is on a caustic and one of the radii of curvature of the wave front, say ρ_2, vanishes there. This necessitates the assumption that the amplitude A_o becomes infinite in such a way that the product $A_o \rho_2^{1/2}$ is proportional to the amplitude at P_1 computed from the previous formulas. The reciprocity principle dictates that the proportionality factor is the same function of the physical parameters as the diffraction coefficient D appearing in (3.64). The complete expression for the field at P in terms of that at P_o is finally written

$$u_d(P) = A_i(P_o) D(P_o) D(P_1) \sqrt{\frac{d\sigma(P_o)}{d\sigma(P_1)}} \cdot \sqrt{\frac{\rho_1}{s_2(\rho_1 + s_2)}} \exp\left\{ ik\left[\phi_i(P_o) + s_1 + s_2\right] - \int_0^{s_1} \alpha(s)\, ds \right\} . \tag{3.66}$$

A further modification must be made in (3.66) for surfaces on which the field is required to vanish. In this case, since the surface is a caustic, there must exist

a sort of boundary layer in the neighborhood of the surface, in which the field is much stronger than in the more distant regions, and which will in general consist of a number of different modes, each with its own amplitude and diffraction and decay coefficients, so that the product $D(P_0)D(P_1) \exp\left[-\int_0^{s_1} \alpha(s)\, ds\right]$ will be replaced by a sum of such products, and the amplitudes appearing will be those at some point slightly separated from the surface.

Since the diffraction and decay coefficients depend primarily on the local geometry of the surface, their essential characteristics should be determinable from the solutions of certain canonical problems, and the values so obtained should hold for a reasonably large class of scatterers. The method used to determine these coefficients in the canonical cases (the circular cylinder and the sphere are sufficiently representative for most purposes) is to expand the exact (wave function) solutions in asymptotic series for small wavelength and compare the dominant terms of these expansions with the forms obtained by means of the above theory, a process which is too lengthy to be treated in detail here. In all cases examined so far the essential forms of these terms are in perfect agreement, and it is a simple matter to isolate the diffraction and decay coefficients. For bodies other than the cylinder and sphere, of course, the problem of determining the exact solution and its asymptotic form is by no means simple, and the latter objective for the prolate spheroid will be discussed presently.

Before we proceed to this, however, there are several considerations in the geometric theory which require further comment. As remarked above, the expressions derived so far become infinite, in general, in the neighborhood of the diffracting body. The modification required to permit their use there can be inferred from the behavior of the exact solutions of the cylinder and sphere problems. In obtaining the asymptotic forms of these for a general field point, the Debye expansion is used for the Hankel functions which appear. For a point on or near the surface, however, the arguments of these functions become approximately equal to the index of the dominant one, and the Debye expansion is no longer suitable, but should be replaced by the Hankel expansion, which is valid for this region and remains finite. Since the diffraction coefficients to be used in the geometrical solutions to general problems are proportional to these factors, it follows that to make these solutions hold in the region of the surface, they should be multiplied by the ratio of the two expansions specified. The correction factor for an axial caustic can be handled in a similar, but simpler, manner by writing the exact expression for a general wave function possessing an axial caustic and comparing this with its asymptotic form, which becomes infinite on the axis. The corrected expression for the surface field of the hard spheroid is given along with the general field expressions in the section on results.

The above theory can, with comparative ease, be adapted to vector problems. For the geometric field (i.e. incident and reflected rays) the forms are identical to

those for the scalar case except that the amplitudes are now vectors and the reflection coefficients matrices. For surface rays, each field quantity is resolved into components parallel to the normal and binormal, and these components are assumed to propagate independently according to the same laws which govern a scalar field, each having its own diffraction and decay coefficients. Those of the normal component (which is also normal to the surface) are taken to be the same as for a scalar field which satisfies a Neumann boundary condition, and those of the binormal (which is tangent to the surface) are taken from the scalar Dirichlet case. For an axially symmetric problem, i.e. backscattering from a solid of revolution with incident direction along the axis, this yields a particularly simple expression for the scattered (vector) field in terms of the two scalar solutions, namely (for the electric field)

$$\underline{E}^s = \frac{1}{2}\left(u_D^s - u_N^s\right)\underline{E}^i \tag{3.67}$$

where u_D^s and u_N^s are the scattered scalar fields of the Dirichlet and Neumann problems, respectively. (Compare this with the relation (3.61) for the total scattering coefficients). The complete radiation pattern for the vector spheroid has, to the best of our knowledge, not yet been worked out.

The details of the geometrical theory as it applies to homogeneous, non-absorptive dielectric bodies are discussed at length in the report of Thomas (1962). Here the situation is quite different, in that the diffracted rays are no longer

significant, and instead there are refracted and internally reflected rays to be considered. Since the reflecting surfaces are no longer all convex the possibility of multiple reflections exists and the geometry of the wave fronts becomes much more complicated. For certain wave fonts one radius of curvature becomes infinite, with the result that expressions of the form (3.63) are no longer applicable, and the principles of physical optics and stationary phase must be employed instead. A general discussion of these is given in Silver (1959). The number and variety of rays which pass through any given exterior point depend on the relative permittivity of the body as well as its geometry, and a general discussion of the problem will not be attempted here. The backscattering echo area of a particular spheroid of particular permittivity has been computed by Thomas and compared with experimental values.+ One important characteristic of this type of problem is that there is no longer a well defined resonance region, since there are no appreciable surface waves, whose interference effects are responsible for the large-scale oscillations in the return from conducting bodies when the wavelength is comparable to the body dimension. As a result the optical approach discussed here gives good results over a frequency range extending down virtually to the Rayleigh region

In the preceding account of the geometrical theory for conducting or rigid bodies, little emphasis has been placed on restrictions in the shape of the scatterer. A more careful consideration however reveals at once that since the radii of curvature of the surface are intimately involved in the development and must satisfy

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+ An attempt to check the numerical results has not succeeded to date and the investigation is still in progress. The curve is therefore omitted here but the experimental results are given on p. 203

certain criteria in terms of the wavelength, any restriction placed on the latter implies some limitation on the shape. In the case of a spheroid, this expresses itself in terms of the eccentricity or axial ratio. Thus in the Keller solution described above the use of diffraction and decay coefficients obtained from the sphere problem, where the radius must be large compared to the wavelength, will result in significant error unless the local radii of curvature of the spheroid meet the same requirement everywhere, i.e. unless the eccentricity is sufficiently small. This is borne out by the analytical results to be considered next. In contrast to the situation at low frequencies, where the form of the scatterer is of minimal importance, nearly all of the high-frequency approximations developed here actually involve a combined restriction on frequency and eccentricity.

Another such method which entails a lower bound on the radius of curvature at each point on the surface is that of Fock (1946). We limit ourselves here to a brief description of this theory, since, as noted above, it yields results which are in general equivalent to those produced by the geometrical theory, and since the particular forms for the spheroid problem have apparently not been worked out. Furthermore the immediate answers provided are limited to the surface current or field distribution in the shadow region, from which it is no trivial task to obtain the scattering pattern or cross section.

The basis of Fock's method is the local approximation of the surface in the region of the shadow boundary by a paraboloid (or in two dimension, a parabolic

cylinder.) If the incident field is a plane harmonic wave and the surface is a perfect conductor (or either perfectly hard or perfectly soft in the scalar case) then the solution is characterized in either vector or scalar problems by a scalar wave function which satisfies a Dirichlet or Neumann boundary condition. Let the incident wave propagate in the z direction and write the field quantity ψ which satisfies this boundary condition and the scalar wave equation as

$$\psi = e^{-ikz} U .$$

Then it is physically reasonable that in the vicinity of the shadow curve and for small enough wavelength, the quantity U should vary much more rapidly in the direction normal to the surface than in any tangential direction. Application of these two approximations leads to a parabolic equation in U, whose solutions are essentially Airy integrals, and the field is finally expressed in term of these functions.

As originally formulated, the theory is essentially two dimensional and applies only in the immediate vicinity of the shadow boundary. However, it has been modified and extended (cf. Goodrich, 1959) to apply to three-dimensional convex bodies and to cover the entire shadow region. The modifications entail a factor which accounts for the increase in energy density of the surface field due to the reduction in area as the rear of the body is approached, i.e. the convergence of the geodesic paths, and a continuous comparison of the normal and tangential field components over the shadow region.

3.1.2.3 Asymptotic Theories

The remainder of this section deals with an analytical approach which involves the work of a number of authors and which leans heavily on the asymptotic theory of the solutions of differential equations involving a parameter. Again a complete account is impossible here, but we will give a general outline of the scheme as a whole and the various contributions of the principal investigators, and present the available results for the spheroid problem in their proper context hereafter. The general approach can be characterized as a refinement and extension of the Watson transform methods which were developed originally in connection with the sphere problem and which have been described in detail in the first report of this series. The basis of the original technique was the observation that since the terms in the Mie series are entire functions of the summation index ν in a strip about the real axis, the sum can be rewritten as a contour integral in the complex ν-plane, whose integrand is the general term of the series with an additional factor to provide poles at the proper points on this axis, such that the residues are the terms of the original series. This integrand has a second set of poles, however, which are the zeroes of a Hankel function appearing in the denominator, and all of which lie in the first quadrant of the ν-plane. When the path of integration is deformed so as to enclose these poles instead of those on the real axis, the resulting residue series is found to converge much more rapidly than the original one at high frequencies. One modification of the procedure was given by Sommerfeld (1949), who obtained the

analogous result in the scalar sphere problem by subjecting each radial eigenfunction in the wave-function series to the given boundary condition, thus determining the complex indices directly. A similar procedure was applied to the scalar spheroid problem by Levy and Keller (1959). In this case the summation indices remain integers but the eigenvalues $\lambda_{mn}(c)$ become complex, with distinct sets obtaining in the soft and hard cases. In both spherical and spheroidal geometries, the representation thus derived has a logarithmic singularity which obtains everywhere on one half of the polar axis. The asymptotic theory referred to above is employed in the evaluation of the terms of the new series in the limit of small wavelength. The first term of this asymptotic series is precisely the solution given by the geometric theory in all cases for which the two have been compared, and it is generally conceded that this will always be true.

The Watson transform method was exploited in the cylinder and sphere problems by Deppermann and Franz (1952, 1954) and Franz (1954). In these articles it was shown that the resulting asymptotic series for the field in the shaded region of the surface could be written in a form such that each term might represent the amplitude of a creeping wave launched at the shadow boundary and traversing the surface. The series apparently diverges in the illuminated region, but this difficulty is resolved by splitting off a series whose sum represents the geometric optics contribution, leaving a convergent series which is again interpretable in terms of creeping waves. Furthermore it was found that the analytical solutions thus developed were in good agreement with certain experimental data.

The general method has been formalized by Kazarinoff and Ritt (1959) with the aid of the complex resolvent theory of Sims (1957) and Phillips (1952). It is shown that in any scalar problem in which the scatterer is a level surface in a coordinate system in which the wave equation separates, the field distribution on the surface can be represented by a contour integral, which can then be evaluated in terms of its residues, at least in the shadow region, by means of Langer's asymptotic theory of solutions of differential equations with turning points (see Langer, 1935). If the problem is axially symmetric, the integrand involves only the product of the radial and angular resolvent Green's functions, each of which has its own set of poles. In the usual type of problem these two sets of poles are separated by the contour, which can in general be closed in such a way as to include either set, at least for a certain range of the angular coordinate of the observation point. Inclusion of the poles of the angular Green's function produces the Mie series (or its non-spherical analog), which converges very slowly at high frequencies. On the other hand those of the radial Green's function yield the rapidly convergent series referred to above. This is the series derived by Kazarinoff and Ritt for the case of a rigid, not-too-thin prolate spheroid struck by a plane scalar wave in the axis of symmetry. Under the given restriction on eccentricity ($\xi_0 = 1 + \epsilon$, $\epsilon > 0$), the asymptotic theory of Langer is applicable and the residues are expressed in terms of Airy integrals or related functions. The results are valid over the entire shadow region of the surface, and a suitable rearrangement of the series permits an interpretation in terms

of creeping waves and a comparison with the results of the geometric theory of Levy and Keller. The first two terms of the residue series are in agreement with this theory, and the third term exhibits a dependence on the radius of curvature at the tip which indicates that the geometric theory is not accurate if this quantity is too small. The details of the analysis are, needless to say, rather involved, and only the final results are presented in the present work.

If the spheroid is long and thin, i.e. $ka >> 1$ and $kb^2/a << 1$, the initial part of the above procedure is still valid. The field distribution on the surface can still be expressed as a contour integral which is evaluated in terms of the residues at the poles of the radial Green's function. The previous asymptotic developments, however, are no longer applicable, and an alternative theory must be used in computing the residues. The solution has been worked out for symmetrical point-source excitation and either standard boundary condition by Goodrich and Kazarinoff (1963). The asymptotic theory employed was developed by McKelvey (1959) and involves Whittaker (or parabolic cylinder) functions in place of the Airy function of the previous solution. This ultimately yields expressions for the surface distribution of the field or its normal derivative in the form of double series, with distinct forms applying in the regions of the shadow boundary and the shaded tip for each boundary condition (see Sec. 4.1.8). Each term in any of these series can be interpreted as a wave whose phase is associated with a certain geodesic path length on the surface and whose amplitude depends in a somewhat complicated manner on

the shape of the surface and the number of times the wave has passed through a tip. The general character of these waves lies somewhere between that of the creeping waves exhibited by the fat spheroid or sphere and that of the traveling waves which are associated with long thin bodies. This appears reasonable enough since with the specified eccentricity and wavelength the spheroid is indeed a long thin body, i.e. the curvature of the geodesic paths along the sides is relatively small and the tips are correspondingly sharp. Accordingly the amplitude decay rate along the sides is no longer an exponential but instead a slowly varying function of η, while at each tip there is either a reflection or transmission through the pole, characterized by the usual phase shift predicted by the geometric theory, and a sharp drop in amplitude due to radiation. The specific form of the decay rate along the sides suggests that the waves are propagating as spherical waves originating at the tips rather than as cylindrical surface waves. In the transition region between the neighborhoods of the shadow boundary and the tip, the formulas become more complicated, and no complete physical interpretation has been attempted.

3.2 ECCENTRICITY-RESTRICTED APPROXIMATIONS

We turn our attention next to certain approximate analytical results which depend fundamentally on assumptions restricting the shape of the scatterer, i.e. the eccentricity of the spheroid. We may divide these solutions into two rather distinct classes. In the first the eccentricity restriction is applied to the forms obtained via the exact (wave function) formulation and the resulting simplification

provides considerable insight into physical phenomena which in the general case are both inherently more complex and masked by the opacity of the representation. In the second, the restriction on the shape of the body is used as a point of departure and thus characterizes the whole solution implicitly. In either case the frequency may not be completely arbitrary, since any of the basic techniques imposes at least some practical limitation, but in each of these solutions the permissible range of frequencies is much wider than that of the eccentricities.

3.2.1 Large Eccentricity

The primordial example of a solution in the first class for a highly eccentric spheroid is the previously cited work originated by Abraham (1898) and extended and refined by Page and Adams (1938), Ryder (1942) and Page (1944). The method used has been described earlier (Section 2.2) and we consider here only certain qualitative features of the results. In addition to investigating the free oscillations of the general prolate spheroid, these authors consider the case of a thin conducting spheroid struck broadside by a wave with electric vector parallel to the major axis. The incident field is assumed to be either instantaneously uniform or a spheroidal function of the angular coordinate with arbitrary index. The plane wave is easily expressed as a series of these functions, and the uniform field can be considered as a degenerate form, i.e. function of index zero.

For the limiting case of the thin rod of length 2F, an incident wave consisting of the nth 'harmonic' alone produces a well defined resonance at a frequency such

that $c \equiv kF \equiv n\pi/2$, for which the induced current in the rod is sinusoidal and much larger than at neighboring frequencies. There are n+1 nodes, counting those at the ends, and the current is exactly is phase with the incident field. As the eccentricity is decreased, i.e. the rod is transformed into an increasingly thick spheroid, the resonance becomes less well defined. The frequency at which the current is maximum decreases as the thickness is increased, and the current leads the field in phase by an increasing amount. The current at resonance is still sinusoidal, but the rate at which it drops off as the frequency departs from the value at resonance becomes lower. For a spheroid of given (large) eccentricity at a frequency below the resonant value, the current still has a sinusoidal character but it leads the field in phase by a substantial amount and the loops near the center of the body are larger than those near the ends. As the frequency is increased above resonance, the nodes move toward the center and the current becomes vanishingly small in an ever-increasing region about each end, and the current lags behind the field by an increasing phase angle.

The situation is of course much more complicated when the incident field consists of something other than a single harmonic, but the general case can be analyzed by means of the techniques used in these articles and the phenomenological elements described should assist in the overall understanding of the problem. Expressions for the scattered fields under certain excitations are given in Sec. 4.1.9.

Another approximate result which is useful for thin spheroids of sufficient length at certain aspects is that afforded by traveling wave theory. The derivation of this is now standard text-book material (e.g. Kraus, 1950) and since it is not characterized by the precise form of the body, we will not dwell on it here. The resulting formula for the cross section as a function of aspect is given by Siegel (1959) and recorded in the Table. It is difficult to tell exactly how the accuracy of this result deteriorates as the length of the spheroid (in wavelengths) or its eccentricity is decreased, but the data given by Siegel (Fig. 26) show good agreement with experimental results in the region where the contribution is largest, which is in general some 18-30^{o} off nose, for a spheroid of axis ratio 10:1 and length 4λ, and it is clear from the nature of the derivation that the results should be even better for longer and thinner bodies.

2.2 Small Eccentricity

At the opposite extreme in the shape parameter range for the prolate spheroid, the body is of course very like a sphere, and the obvious line of approach to the determination of its scattering properties is via a shape perturbation applied to the classical sphere solution. In this manner an approximate solution should be obtainable, without the encumbrance of the spheroidal functions or even their natural coordinates, which is restricted in frequency only in the sense that the Mie series is, and whose accuracy must improve as the eccentricity becomes smaller. This type of analysis has been carried out by Mushiake (1956) for the scattering of a plane

electromagnetic wave by a conducting spheroid with arbitrary directions of incidence and polarization. The corresponding forms for the scalar problem of a rigid spheroid with symmetrical incidence are given in an unpublished Radiation Laboratory Memo by Sl ator and Ullman (1959). The scalar solution for arbitrary incidence should be easily derivable from the vector forms given by Mushiake, but the explicit expressions have not been written out.

In any case the first step is to write the expansions of the incident and scattered fields in series of spherical (vector or scalar) wave functions. In the vector problem, the spherical vector wave functions of Hansen are employed, and the scattered field expansion has the same general form as in the sphere problem, though the incident field expansion, since the direction of propagation can no longer in full generality be restricted to the z-axis, is more complicated. We can, however, restrict the propagation vector to the xz-plane, so that its direction is specified by a single angle α, and assuming the usual time dependence $e^{-i\omega t}$, the expansions of the incident and scattered electric fields for the two fundamental polarizations ($\underline{E}^i$ perpendicular or parallel to the y axis) take the general forms

$$\underline{E}_{\perp}^{i,s} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ A_{omn}^{i,s} \underline{M}_{omn}^{(1,2)} + B_{emn}^{i,s} \underline{N}_{emn}^{(1,2)} \right\} \tag{3.68}$$

$$\underline{E}_{||}^{i,s} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ A_{emn}^{i,s} \underline{M}_{emn}^{(1,2)} + B_{omn}^{i,s} \underline{N}_{omn}^{(1,2)} \right\} \tag{3.69}$$

Here the $\underline{M}$ and $\underline{N}$ vectors are the standard spherical vector wave functions described in the first report of this series, the superscripts 1 and 2 pertaining to the incident and scattered fields respectively. The coefficients A^i and B^i which define the incident fields are found by the familiar procedure of expanding the vector functions and utilizing the orthogonality of the angular functions involved. Determination of the scattered field coefficients A^s, B^s is somewhat harder, though the scheme is fairly straightforward. The general surface of revolution symmetric about the z-axis can be specified in spherical coordinates by giving the radius r as a function of θ, and for a general spheroid the relation can be written

$$r = f(\theta) \equiv a\left[1 - \nu \sin^2 \theta\right]^{-\frac{1}{2}} \tag{3.70}$$

where, for convenience, we have defined the quantity

$$\nu \equiv (b^2 - a^2)/b^2 \tag{3.71}$$

with a and b as defined earlier. (It should be noted that for $\nu < 0$ the spheroid is prolate, and for $\nu > 0$ it is oblate.) The expression (3.70) must now be inserted in the two equations which obtain on the conducting surface and which in this case have the form

$$E_\phi = 0 = E_\theta + \frac{1}{r}\frac{df}{d\theta} E_r \quad \text{on} \quad r = f(\theta), \tag{3.72}$$

E_r, E_θ, E_ϕ being the components of the total electric field $\underline{E} = \underline{E}^i + \underline{E}^s$, for either polarization, which are obtained by using the explicit forms of the vector wave

functions in (3.68) and (3.69). Needless to say, with the radial variable dependent on the angular one, the orthogonality relations which simplify the solution in the sphere problem are destroyed utterly, and it is no longer possible to obtain an explicit expression for each coefficient of the scattered field in terms of the corresponding pair in the incident field expansion. The equations become manageable only if all the radial functions are replaced by approximate expressions correct to the first order in ν, viz.

$$r \simeq a(1+\frac{\nu}{2}\sin^2\theta), \quad j_n(kr) \simeq j_n(ka)+\frac{\nu}{2}\, ka\, j_n'(ka)\sin^2\theta, \text{ etc.}$$

and the validity of the subsequent forms is thus limited to cases where $|\nu|^2 << 1$, which is the characteristic feature of the perturbation technique. The desired solutions are finally obtained via a process of multiplying the boundary equations by suitable angular functions, integrating over the interval $0 \leqslant \theta \leqslant \pi$, and combining the results in such a way as to yield expressions for each A^s, B^s containing several pairs of the A^i, B^i. The scattered fields are then given by (3.68) and (3.69) in the form of rather complicated double summations. The sphere solution can of course be split off and the first order correction term due to the shape perturbation isolated. Fortunately there is a considerable simplification in the results for the special directions of incidence and observation. The essential results are tabulated hereafter, and certain curves computed for particular cases are also reproduced (see Figs. 21, 22, 34). A complete discussion of the accuracy and applicability range has not been

given, but a comparison with experimental data (Fig. 34) shows reasonably good agreement for a spheroid of axis ratio $b/a = .8$ over a wide angular range.

The scalar problem is handled by means of the same general technique, though the analysis is of course considerably simpler, especially as carried out in the aforementioned memo under the restriction of symmetrical incidence. Here the incident field is simply

$$\phi^i = e^{-ikr\cos\theta} = \sum_n (-i)^n (2n+1) j_n(kr) P_n(\cos\theta) \tag{3.73}$$

and the scattered field has an expansion of the form

$$\phi^s = \sum_n A_n h_n^{(1)}(kr) P_n(\cos\theta). \tag{3.74}$$

If the total field is $\phi \equiv \phi^i + \phi^s$, the boundary condition on the rigid surface specified by $r = f(\theta)$ is

$$f^2 \frac{\partial\phi}{\partial r} - f' \frac{\partial\phi}{\partial\theta} = 0 . \tag{3.75}$$

When the field expansions (3.73) and (3.74) and the perturbation forms given above are introduced in (3.75), the angular dependence can be incorporated entirely in the arguments of three Legendre polynomials with different indices, and the orthogonality relation can then be used to find a finite and relatively simple expression for the general coefficient A_n. Again the correction term is easily separated from the sphere result, but as in the vector case the former is more difficult to compute than the latter.

It would presumably be possible in both the vector and scalar cases to increase the accuracy of the solution, or extend the range of applicability with given accuracy, by retaining all terms in ν^2 throughout the derivations. For the case of general incidence however, and particularly in the vector problem, the amount of labor involved would be formidable, and even in the simpler cases it would not be small.

3.3 APPROXIMATIONS FOR WEAK SCATTERERS

There remains to be considered one class of approximate solutions whose derivations are based on assumptions restricting the properties of the media involved. Technically speaking, of course, the case of a perfect conductor in a non-conducting medium might fall into this class at least as a limiting form, but this case is at once so distinctive and so important as to warrant the separate treatment given it. The problem we now deal with lies at the other extreme in the material parameter range, i.e. where the propagation constant in the interior of the scatterer differs very little from that in the surrounding medium, and the phase shift suffered by the incident wave is thus relatively small. Under these conditions the scatterer is termed weak and can be treated essentially as a perturbation of the medium.

The natural representation of the scattered field in this type of problem is an integral over the volume of the scatterer which is obtainable via Green's theorem and whose integrand involves the Green's function and the internal field. This expression itself is rigorous but since the exact form of the internal field is not

known in general, some approximation must be introduced, and this accordingly characterizes the result. In the most elementary application of the method, the internal field is taken to be exactly what the incident field would be in the absence of the scatterer. This yields what is known as the Rayleigh-Gans-Born approximation, which is of rather limited utility in the type of problem of interest here and which we will not consider further. Instead we will deal with several refinements which give considerably improved results over a wider range of the parameters.

3.3.1. Scalar Case

The first of these was developed by Montroll and Hart (1951) and applied to the scalar problem of a homogeneous spheroid of material properties not too different from the surrounding medium, struck by a plane wave at an arbitrary angle of incidence. The integral expression for the scattered field is obtained by considering the entire space as a medium of variable propagation function $k(\underline{r})$. The scalar wave equation is thus

$$\left[\nabla^2 + k^2(\underline{r})\right] \psi = 0$$

where ψ is the total field, equal to the sum $\psi^i + \psi^s$ of incident and scattered fields, and if the spheroid occupies the volume V, the function $k(\underline{r})$ is specified as

$$k(\underline{r}) = k_o \text{ at all points outside V}$$

$$= k_1 \text{ at all points inside V.}$$

The boundary conditions to be satisfied are

a) continuity of ψ and its first derivative at the boundary of the spheroid,

b) boundedness of these quantities at infinity.

Assuming unit amplitude, the incident wave can be written

$$\psi^i = e^{ik_o \hat{r}_o \cdot \underline{r}}$$

where $\hat{r}_o$ is the unit vector in the incident direction. The wave equation can be written

$$\left[\nabla^2 + k_o^2\right] \psi^s = (k_o^2 - k^2(\underline{r}))\psi$$

and if we consider this as an inhomogeneous equation in the unknown function ψ^s, the solution can be expressed in integral form, using the free space Green's function (2.19), as

$$\psi^s(\underline{r}) = -\frac{1}{4\pi}\int \frac{e^{ik_o|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\left[k_o^2 - k^2(\underline{r}')\right]\psi(\underline{r}')\,dv\,,$$

where the integration covers the entire space. The bracketed quantity in the integrand, however, vanishes at all points exterior to the spheroid, so that the expression can actually be written

$$\psi^s(\underline{r}) = \frac{\left(k_1^2 - k_o^2\right)}{4\pi}\int_V \frac{e^{ik_o|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\psi(\underline{r}')\,dv',$$

or at large distance r from the scatterer,

$$\psi^s(\underline{r}) \simeq \frac{\left(k_1^2 - k_o^2\right)}{4\pi\, r}e^{ik_o r}\int_V e^{-ik_o\hat{r}\cdot\underline{r}'}\psi(\underline{r}')\,dv'. \qquad (3.76)$$

There remains the problem of ascertaining or approximating the field $\psi(\underline{r})$ interior to the scatterer. Here Montroll and Hart make the assumption that for a long thin spheroid the interior field should be approximately equal to that in an infinite cylinder of diameter equal to the minor axis of the spheroid and material properties the same. The internal field of the cylinder can be determined rigorously under the assumption of continuity of the normal particle velocity at the surface, which involves the ratio of the densities of the two media as another essentially independent parameter. The solution has the form of an infinite series of cylindrical functions, however, and in view of the error already introduced by the assumption of the cylindrical field for the spheroid problem, the use of the exact expression is hardly warranted. Instead it is observed that if the coefficients in the cylinder result are altered in a manner which, in the case where the interior and exterior densities and propagation constants are nearly equal, changes their values very little, the series can be summed, and when the resulting exponentials are substituted in the integrand of (3.76) the integrations can be carried out in closed form.

The approximate expression thus obtained for the far-zone field scattered by a thin, tenuous spheroid (See Sec. 4.1.12 p. 170) is not asymptotic to the exact solution in any one parameter alone, since there are three essentially independent approximations involved. As the density and propagation constant of the spheroid's interior approach those of the surrounding medium, the approximate solution is

asymptotic to the exact, at least in the sense that for both the scattered field approaches zero. The accuracy of the approximation should improve, in some range at least, as either of the ratios approaches unity, or as the axial ratio a/b of the spheroid becomes larger, but the details of these variations are not given. It should be noted that the frequency is not explicitly involved in any of the approximating assumptions, except as it appears in the definition of the propagation constant. The validity of the result should thus be relatively insensitive to the frequency, though some variation is almost certainly present.

Another approximate scalar result for weak scatterers has been given by Greenberg (1960). This is based on the Born series solution for the Schrödinger equation under the conditions that the range of the potential, i.e. dimension of the scatterer, is large compared to the wavelength and the energy of the potential is small compared to that of the incident field. If, in addition, the scattering angle is small, then the Born series is easily summed and the scattered amplitude is given in terms of a triple integral involving the potential (see Schiff, 1956). For a square well complex potential of spheroidal form the integrations have been carried out by Greenberg to yield an expression for the total scattering cross section, which is proportional to the imaginary part of the forward scattering amplitude. The result is listed in Sec. 4.1.12.

2. Vector Case

Certain vector problems analogous to the scalar ones considered above are also capable of formulation in terms of an integral equation. If the two media are assumed to have the same permeability and the dielectric constants are ϵ and ϵ_o for the interior and exterior respectively, and if a plane wave with propagation vector $k_o\hat{r}_o$ and constant amplitude vector $\underline{E}^i$ (perpendicular to $\hat{r}_o$) strikes the spheroid, the integral equation for the total electric field $\underline{E}(\underline{r})$ can be written (suppressing the usual time dependence $e^{-i\omega t}$)

$$\underline{E}(\underline{r}) = \frac{(\epsilon - \epsilon_o)}{4\pi\epsilon_o} \nabla \wedge \nabla \wedge \int_V \frac{e^{ik_o\rho}}{\rho} \underline{E}(\underline{r}')\, dv' + e^{ik_o\hat{r}_o \cdot \underline{r}}\, \underline{E}^i. \qquad (3.77)$$

where $\rho = |\underline{r} - \underline{r}'|$ and the integration in the variable $\underline{r}'$ covers the interior of the spheroid as before. The essential problem is again the choice or determination of an approximation to the internal field $\underline{E}(\underline{r}')$. Two independent attacks on this problem exist in the literature and will be outlined here. The first was carried out by Shatilov (1960). His basic assumption is that the amplitude of the internal field is just that which would be produced by a uniform external field, while the phase is that of the incident field. The explicit form of the amplitude is obtained from (3.77) by taking the field point $\underline{r}$ inside V and letting $k_o = 0$, i.e. taking only the first term in the expansions of the exponentials. The amplitude is thus

$$\underline{E}_{io} = \underline{E}^i + \frac{(\epsilon-\epsilon_o)}{4\pi\epsilon} \nabla\wedge\nabla\wedge \left[\underline{E}^i \int_V \frac{1}{\rho}\, dv' \right]$$

and the entire internal field is

$$\underline{E}_i = \underline{E}_{io}\, e^{ik_o \hat{r}_o \cdot \underline{r}} \tag{3.79}$$

This assumption yields in effect a refinement of the Rayleigh-Gans-Born approximation in the domain of the material parameters, but it introduces at the same time a serious restriction on the frequency, so that the applicability of the results is necessarily limited to the Rayleigh region. For the scattered field in the far zone, the formula (3.77) yields, after some manipulation, the expression

$$\underline{E}^s(\underline{r}) = \frac{k_o^2(\epsilon-\epsilon_o)}{4\pi\epsilon_o} \int_V \hat{\rho}\wedge(\hat{\rho}\wedge\underline{E}_i)\frac{e^{ik_o\rho}}{\rho}\, dv' \tag{3.80}$$

with $\hat{\rho} \equiv (\underline{r}-\underline{r}')/\rho$. By virtue of (3.79) and the far-field condition this can be further simplified to the form

$$\underline{E}^s(\underline{r}) = -\frac{k_o^2(\epsilon-\epsilon_o)}{4\pi\epsilon_o}\underline{E}_{io}\frac{e^{ik_o\hat{r}_o\cdot\underline{r}}}{|\underline{r}|} \int_V e^{ik_o(\hat{\rho}-\hat{r}_o)\cdot\underline{r}'}\, dv' \tag{3.81}$$

(note that in this approximation the propagation constant k inside the spheroid is the same as k_o outside.) The integral can be evaluated explicitly for the spheroidal scatterer with arbitrary directions of incidence and observation.

If the geometry of the setup is as shown in Fig. 3, where the incident and observation directions are separated by an arbitrary angle β, and the symmetry axis of the spheroid makes an angle α with the bisector of the complement of β, the plane of the angle α being unrestricted, the field can be written finally

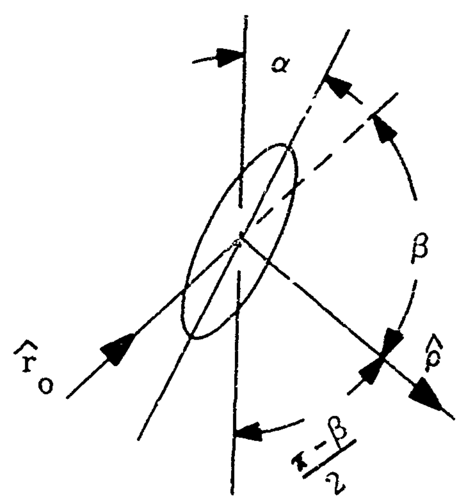


FIG. 3

$$\underline{\underline{E}}^S(\underline{r}) = -\underline{E}_{io} \frac{(\epsilon-\epsilon_o)}{4\pi\epsilon_o} k_o^2 \, e^{ik_o \hat{r}_o \cdot \underline{r}} \cdot V f(q) \tag{3.82}$$

where V is the volume of the spheroid $\xi = \xi_o \equiv a/\sqrt{a^2-b^2}$

$$f(q) \equiv 3q^{-3}(\sin q - q\cos q)$$

and

$$q \equiv \frac{2k_o a}{\xi_o}\sqrt{\xi_o^2 - \sin^2\alpha} \cdot \sin\beta/2.$$

(Compare this expression for the scattered field with the form given by Siegel (1959), p. 72 of this report, based on the dipole approximation.)

The second attack on the weak-scattering problem is that of Ikeda (1963), again based on the integral formulation (3.77), but employing an expansion technique which yields a more general result. Instead of the a priori assumption of the interior field used by Shatilov, Ikeda assumes an expansion of the electric field at the general point $\underline{r}$ in powers of the (small) quantity $(\epsilon_0-\epsilon)/\epsilon$, i.e. a power series in terms of ϵ_0 about the value ϵ, which is written

$$\underline{E}(\underline{r}) = \sum_{n=0}^{\infty} \left(\frac{\epsilon_0-\epsilon}{\epsilon}\right)^n \underline{E}_n(\underline{r}) . \tag{3.83}$$

Also the exterior propagation constant k_0 is written in terms of the interior value k as

$$k_0 = k(\epsilon_0/\epsilon)^{1/2}$$

and when these expressions are substituted in (3.77) and the coefficients of like powers of the argument are equated, there results a set of equations which express each vector $\underline{E}_n(\underline{r})$ explicitly in terms of the preceding ones and the incident field vector $\underline{E}^i$, and the expansion (3.83) can thus, in principle at least, be carried out to any degree desired. Since this expression is valid everywhere, it can be used for the exterior field $\underline{E}_i$ in (3.80), and the scattered field is thus given explicitly as a power series in ϵ_0.

This technique is used by Ikeda to determine the cross-polarization elements of the scattering matrix to the first-order approximation. The remaining elements

are determined to the zero-order approximation. The latter result, in which the cross-polarization elements of the matrix vanish, is comparable to the Rayleigh-Gans-Born approximation, offering a slight advantage in that here the true interior propagation constant k appears in the internal field expression instead of the exterior value k_o. It should be noted that there is no absolute or implicit restriction on either eccentricity or frequency involved in this method, though from the nature of the forms involved, some of which are tabulated in the next section, it is to be expected that results of a given accuracy will be more easily obtained at lower frequencies.

IV

RESULTS

The foregoing discussion of the analytical solutions of the spheroid problem has been kept reasonably free of detailed and specific formulas, on the theory that the number and complexity of the pertinent forms would, if included, tend more to obscure than to elucidate the reasoning involved. In the first part of the following are tabulated the principal end results of the various analyses, together with references to the sources and pertinent sections of the preceding text and any available information on accuracy, range of validity, etc. The second part is a compilation of quantitative data including the majority of the curves or points, both theoretical and experimental, obtained and published by the principal investigators of the problem to date.

4.1 TABULATION OF FORMULAS

1. Exact Scalar Solutions (see Sec. 2.2.1, pp 25-31, also Spence and Granger, 1961).

The specialized forms of the fundamental scalar solutions for source point in the axis of symmetry are as follows:

Eq. (2.23) becomes

$$G(\underline{r}, \underline{r}_1) = \frac{ik}{2\pi} \sum_{n=0}^{\infty} \frac{1}{N_{on}} S_{on}(c, 1) S_{on}(c, \eta) R^{(3)}_{on}(c, \xi_>) \left[R^{(1)}_{on}(c, \xi_<) - C_{on} R^{(3)}_{on}(c, \xi_<) \right]$$

Eq. (2.24) becomes

$$G_{\infty}(\xi,\eta;\pi) = 2\sum_{n=0}^{\infty} \frac{i^n}{N_{on}} S_{on}(c,1)\, S_{on}(c,\eta) \left[R_{on}^{(1)}(c,\xi) - C_{on} R_{on}^{(3)}(c,\xi) \right] .$$

If these are further specialized by putting the observation point in the far zone, they become respectively

$$G(\underline{r},\underline{r}_1) \simeq -\frac{ie^{ikr}}{2\pi r} \sum_{n=0}^{\infty} \frac{(-1)^n}{N_{on}} S_{on}(c,1)\, S_{on}(c,\cos\theta) \left[R_{on}^{(1)}(c,\xi_1) - C_{on} R_{on}^{(3)}(c,\xi_1) \right]$$

$$\text{and } G_{\infty}(r,\theta;\pi) \simeq \frac{2}{kr} \sum_{n=0}^{\infty} \frac{i^n}{N_{on}} S_{on}(c,1) S_{on}(c,\cos\theta) \left[\cos\left(kr - \frac{n+1}{2}\pi\right) - C_{on} e^{i\left(kr - \frac{n+1}{2}\pi\right)} \right]$$

and in this case eq. (2.25) becomes

$$G_0(r,\theta;\xi_1,1) \simeq \frac{e^{ikr}}{\pi r} \sum_{n=0}^{\infty} \frac{(-i)^n}{N_{on}} S_{on}(c,1) S_{on}(c,\cos\theta)\, R_{on}^{(1)}(c,\xi_1).$$

In these formulas the quantity C_{on} is as given in (2.23), i.e.

$$C_{on} \equiv \frac{\alpha R_{on}^{(1)}(c,\xi_0) + \beta \frac{\partial}{\partial n} R_{on}^{(1)}(c,\xi_0)}{\alpha R_{on}^{(3)}(c,\xi_0) + \beta \frac{\partial}{\partial n} R_{on}^{(3)}(c,\xi_0)}$$

with α, β as in (2.22).

In the case $\alpha = 0$, $\beta = 1$ (scattering of sound by a hard spheroid) a number of calculations of scattered far field have been carried out by Spence and Granger (1951)

for plane wave incidence (G_∞). Their results appear in Figs. 4-8.* In addition some nose-on back scattering cross sections have been computed by Siegel et al(1956) and Crispin et al (1963). Their results appear in Fig. 9.

2. Axial Dipole Solution (see Sec. 2.2.2, p.32, also Hatcher and Leitner, 1954).

The asymptotic form of eq. (2.41), which gives the far zone radiation pattern with the dipole at the tip of the spheroid is

$$H_\phi^T(\theta) \simeq \frac{-2ikp}{F(\xi_0^2-1)} \sum_{n=0}^{\infty} \frac{(-1)^n S_{1n}(c, \cos\theta)}{\rho_{1n} N_{1n}\left[\frac{\partial}{\partial\xi}\left(\sqrt{\xi^2-1}\, R_{1n}^{(3)}(c,\xi)\right)\right]_{\xi_0}}$$

where, as in (2.41), p is the dipole strength and ρ_{1n} is the normalizing factor of the radial functions, as defined in (2.14). Radiation patterns for a dipole on a spheroid have been calculated by Hatcher and Leitner (1954). Their results appear in Figures 10-12. Belkina (1957) has also calculated some radiation patterns of an axially symmetric dipole located on the surface of a spheroid. Her results are presented in Figure 42.

3. General Vector Solution (See Sec. 2.3, p. 42, also Siegel et al, 1956).

The scattered electric field in the far zone produced by a conducting spheroid struck by a plane wave propagating parallel to the major axis is given by eq. (2.55)

---

*Note that the scattering patterns as originally published omitted the units, i.e. the ordinates plotted are actually values of the quantity $f(\theta,\phi)/a$, where $f(\theta,\phi) = re^{-ikr} \cdot \lim_{y\to\infty} \psi^s$, ($\psi^s$ scattered field) and a = semi-major axis.

as a function of the angular coordinates η, ϕ of the observation point. From this the radar cross section is easily found to be

$$\sigma(\eta,\phi) = \frac{4\pi}{\left|\underline{E}^i\right|^2}\left\{\sin^2\phi\left|\sum_{n=0}^{\infty} i^n A_n^x S_{on}(c,\eta)\right|^2 + \right.$$

$$\left. + \cos^2\phi\left|\sum_{n=0}^{\infty} i^n\left[A_n^x \eta S_{on}(c,\eta) - i A_n^z \sqrt{1-\eta^2}\, S_{1n}(c,\eta)\right]\right|^2\right\}$$

and for backscattering this reduces to

$$\sigma = \frac{4\pi}{\left|\underline{E}^i\right|^2}\left|\sum_{n=0}^{\infty} i^n A_n^x S_{on}(c,1)\right|^2$$

The results of a numerical computation of electromagnetic backscattering cross section (Siegel et al, 1956) are presented in Figure 24, Section 4.3. The coefficients A_n^x, A_n^z are found as indicated in the text by solving the linear equations (2.52), (2.53). If these are truncated after the fourth term, as in the computations of Siegel et al (1956), the solutions may be written in determinantal form as follows:

The A_n^x have the forms

$$A_o^x = \frac{aE^i}{G}\begin{vmatrix} B_{00} & O & D_{01} & D_{03} \\ B_{22} & C_{22} & D_{21} & D_{23} \\ U_{10}+U_{12} & V_{12} & W_{11} & W_{13} \\ U_{30}+U_{32} & V_{32} & W_{31} & W_{33} \end{vmatrix}$$

$$A_1^x = \frac{a E^i}{H} \begin{vmatrix} B_{11} & O & D_{10} & D_{12} \\ B_{33} & C_{33} & D_{30} & D_{32} \\ U_{01}+U_{03} & V_{03} & W_{00} & W_{02} \\ U_{21}+U_{23} & V_{23} & W_{20} & W_{22} \end{vmatrix}$$

$$A_2^x = \frac{a E^i}{G} \begin{vmatrix} C_{00} & B_{00} & D_{01} & D_{03} \\ O & B_{22} & D_{21} & D_{23} \\ V_{10} & U_{10}+U_{12} & W_{11} & W_{13} \\ V_{30} & U_{30}+U_{32} & W_{31} & W_{33} \end{vmatrix}$$

$$A_3^x = \frac{a E^i}{H} \begin{vmatrix} C_{11} & B_{11} & D_{10} & D_{12} \\ O & B_{33} & D_{30} & D_{32} \\ V_{01} & U_{01}+U_{03} & W_{00} & W_{02} \\ V_{21} & U_{21}+U_{23} & W_{20} & W_{22} \end{vmatrix}$$

and similar expressions obtain for the A_n^z. The denominators G and H are given by the expressions

$$
G = \begin{vmatrix} C_{00} & O & D_{01} & D_{03} \\ O & C_{22} & D_{21} & D_{23} \\ V_{10} & V_{12} & W_{11} & W_{13} \\ V_{30} & V_{32} & W_{31} & W_{33} \end{vmatrix}
$$

$$
H = \begin{vmatrix} C_{11} & O & D_{10} & D_{12} \\ O & C_{33} & D_{30} & D_{32} \\ V_{01} & V_{03} & W_{00} & W_{02} \\ V_{21} & V_{23} & W_{20} & W_{22} \end{vmatrix}
$$

the elements B_{rn}, C_{rn}, - - - - W_{rn} in the above are defined in eqs. (2.54). The integrals which appear there can be expressed directly in terms of the spheroidal coefficients (cf. Sec. 2.1.2) as follows using the Kronecker delta, δ_{rn}, and the parity modulus, μ_{rn}, which are defined respectively as

$$
\delta_{rn} = \begin{cases} 0 \text{ for } r \neq n \\ 1 \text{ for } r = n \end{cases}, \quad \mu_{rn} = \begin{cases} 0 \text{ for } r+n \text{ odd} \\ 1 \text{ for } r+n \text{ even.} \end{cases}
$$

$$\int_{-1}^{1} S_{on} S_{or}\, d\eta = 2\delta_{rn} \sum_{k=0}^{\infty} \mu_{rk} \frac{(d_k^{on})^2}{2k+1} \equiv N_{on}$$

$$\int_{-1}^{1} \frac{\eta}{\sqrt{1-\eta^2}} S_{1n} S_{or}\, d\eta = -2\, \mu_{r(n+1)} \left[\sum_{k=0}^{\infty} \mu_{nk} \frac{k+1}{2k+3} d_k^{1n}\; d_{k+1}^{or} \right.$$

$$\left. + \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \mu_{rk} \mu_{nj}\, d_k^{or}\; d_j^{1n} \right]$$

$$\int_{-1}^{1} \eta S_{on} S_{or}\, \overline{d\eta} = 2\mu_{r(n+1)} \sum_{k=0}^{\infty} \frac{(k+1)}{(2k+1)(2k+3)} \left(d_k^{on}\; d_{k+1}^{or} + d_{k+1}^{on}\; d_k^{or}\right)$$

$$\int_{-1}^{1} (1-\eta^2) \frac{dS_{on}}{d\eta} S_{or}\, d\eta = 2\, \mu_{r(n+1)} \sum_{k=0}^{\infty} \frac{k(k+1) d_k^{on}}{(2k-1)(2k+1)(2k+3)} \left[(2k+3) d_{k-1}^{or} - (2k-1) d_{k+1}^{or} \right]$$

$$\int_{-1}^{1} \sqrt{1-\eta^2}\; S_{1n} S_{or}\, d\eta = 2\mu_{rn} \sum_{k=0}^{\infty} \frac{(k+1)}{(2k+1)(2k+3)} \left[k\, d_{k-1}^{1n}\; d_{k+1}^{or} - (k+2)\, d_k^{1n}\; d_k^{or} \right]$$

$$\int_{-1}^{1} \eta \sqrt{1-\eta^2}\; \frac{d}{d\eta} S_{1n} S_{or}\, d\eta \qquad = 2\mu_{rn} \left\{ \sum_{k=0}^{\infty} \frac{(k+1)^2}{(2k+1)(2k+3)} \left[(k+2)\, d_k^{1n}\; d_k^{or} \right. \right.$$

$$\left. + k\, d_{k-1}^{1n}\; d_{k+1}^{or} \right] + \sum_{k=0}^{\infty} \frac{\mu_{nk}}{(2k+1)}\, d_k^{or} \left[\frac{k(k-1)}{(2k-1)}\, d_{k-2}^{1n} + \right.$$

$$+\frac{(3k^2+5k+1)}{(2k+3)}\, d_k^{1n}\Bigg] + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \mu_{nj}\mu_{nk}\, d_j^{1n}\, d_k^{or}\Bigg\}.^{*}$$

Accuracy of the 4-term result depends inversely on ka and in a more complicated but not so critical manner on ξ_o. At a value $\xi_o = 1.005\,(a/b = 10)$ the result is correct to two significant figures out to ka$\lesssim$5.

4. Rayleigh Series

a. Scalar Case (See Sec. 3.1.1, 61, also Senior, 1960a).

The coefficients $u_n(\eta)$ and $v_n(\eta)$ in the series (3.2) and (3.3) for the far-field amplitude of a soft or hard spheroid struck by a plane scalar wave in the axis of symmetry are given for $\eta = 0$ - - - 5 in the following table. Except where otherwise specified, the argument of all Legendre functions appearing is ξ_o. Primes denote derivatives with respect to the argument.

* Note that $d_k^{mn} = 0$ for $k < 0$.

TABLE I

n	$u_n(\eta)$ (Soft Spheroid)	$v_n(\eta)$ (Hard Spheroid)
0	$(P_o/Q_o)P_o(\eta)$	0
1	$(P_o/Q_o)^2 P_o(\eta)$	0
2	$\frac{1}{9}\frac{P_o}{Q_o}P_2(\eta) + \frac{1}{3}\frac{P_o}{Q_1}P_1(\eta) +$ $\frac{P_o}{Q_o}\left[\left(\frac{P_o}{Q_o}\right)^2 - \frac{1}{3}\frac{P_1}{Q_o} + \frac{2}{9}\right] \cdot P_o(\eta)$	$\frac{1}{3}\frac{P'_1}{Q'_1}P_1(\eta) + \frac{1}{9}\frac{P'_2}{Q'_o}P_o(\eta)$
3	$\frac{1}{9}\left(\frac{P_o}{Q_o}\right)^2 P_2(\eta) + \left(\frac{P_o}{Q_o}\right)^2\left[\left(\frac{P_o}{Q_o}\right)^2 - \frac{2}{3}\frac{P_1}{Q_o} + \frac{1}{3}\right]P_o(\eta)$	0
4	$\frac{1}{525}\frac{P_o}{Q_o}P_4(\eta) + \frac{1}{75}\frac{P_1}{Q_1}P_3(\eta) +$ $\frac{1}{9}\left[\frac{4}{45}\frac{P_2}{Q_2} + \frac{P_o}{Q_o}\left\{\left(\frac{P_o}{Q_o}\right)^2 - \frac{1}{3}\frac{P_1}{Q_o} + \frac{16}{63}\right\}\right]P_2(\eta)$ $+\frac{1}{3}\frac{P_1}{Q_1}\left[\frac{1}{25}\left(\frac{P_3}{P_1} - \frac{Q_3}{Q_1}\right) + \frac{1}{6}\frac{P_o}{Q_1} + \frac{4}{25}\right]P_1(\eta)$ $-\frac{P_o}{Q_o}\left[\frac{1}{75}\frac{P_3}{Q_o} + \frac{P_1}{Q_o}\left\{\left(\frac{P_o}{Q_o}\right)^2 + \frac{1}{27}\frac{Q_2}{Q_o} - \frac{1}{6}\frac{P_1}{Q_o} + \frac{91}{675}\right\}\right.$ $\left.-\left(\frac{P_o}{Q_o}\right)^4 - \frac{4}{9}\left(\frac{P_o}{Q_o}\right)^2 - \frac{2}{75}\right]P_o(\eta)$	$\frac{1}{75}\frac{P'_1}{Q'_1}P_3(\eta) + \frac{1}{81}\left(\frac{4}{5}\frac{P'_2}{Q'_2} + \frac{P'_2}{Q'_o}\right)P_2(\eta)$ $+\frac{1}{75}\frac{P'_1}{Q'_1}\left(\frac{P'_3}{P'_1} - \frac{Q'_3}{Q'_1} + 4\right)P_1(\eta) +$ $\frac{1}{3}\left[\frac{1}{175}\frac{P'_4}{Q'_o} - \frac{1}{27}\frac{P'_2}{Q'_o}\left\{\frac{Q'_2}{Q'_o} + \frac{9}{2}\frac{P'_1}{Q'_o} + \frac{16}{7}\right\}\right]P_o(\eta)$

TABLE I (Cont.)

n	$u_n(\eta)$ (Soft Spheroid)	$v_n(\eta)$ (Hard Spheroid)
5	$\frac{1}{525}\left(\frac{P_o}{Q_o}\right)^2 P_4(\eta)+\frac{1}{9}\left(\frac{P_o}{Q_o}\right)^2\left[\left(\frac{P_o}{Q_o}\right)^2-\frac{2}{3}\frac{P_1}{Q_o}+\frac{23}{63}\right]P_2(\eta)$ $-\frac{1}{27}\left(\frac{P_1}{Q_1}\right)^2 P_1(\eta)-\left(\frac{P_o}{Q_o}\right)^2\left[\frac{2}{75}\frac{P_3}{Q_o}+\frac{4}{3}\frac{P_1}{Q_o}\cdot\right.$ $\cdot\left\{\left(\frac{P_o}{Q_o}\right)^2-\frac{1}{18}\frac{Q_2}{Q_o}-\frac{1}{3}\frac{P_1}{Q_o}+\frac{58}{225}\right\}$ $\left.-\left(\frac{P_o}{Q_o}\right)^4-\frac{5}{9}\left(\frac{P_o}{Q_o}\right)^2-\frac{127}{2025}\right]P_o(\eta)$	$-\frac{1}{27}\left(\frac{P'_1}{Q'_1}\right)^2 P_1(\eta)+\frac{1}{81}\left(\frac{P'_2}{Q'_o}\right)^2 P_o(\eta)$

The backscattering cross section of a hard spheroid of axis ratio 10:1 computed from this series is plotted as a function of ka in Fig. 14. The dependence of the accuracy on the axis ratio has not been thoroughly analyzed.

b. Vector Case (See Sec. 3.1.1, p. 64, also Justice, 1956)

The incident and scattered fields about the spheroid are assumed to be representable as power series of the forms shown in (3.15). In the solution for the conducting spheroid in terms of vector mode functions, the incident field is assumed to propagate parallel to the z-axis with electric vector in the y direction. The first three coefficients in the incident field expansions are then (in rectangular coordinates)

$$\underline{E}_o^i = i_x, \quad \underline{E}_1^i = z\, i_x, \quad \underline{E}_2^i = \frac{z^2}{2}\, i_x$$

$$\underline{H}_o^i = i_y, \quad \underline{H}_1^i = z\, i_y, \quad \underline{H}_2^i = \frac{z^2}{2}\, i_y$$

and those of the scattered electric field are

$$\underline{E}_o^s = -\frac{P_1^1}{Q_1^1}\,\underline{S}_{e_{11}}^{(2)}$$

$$\frac{1}{F}\,\underline{E}_1^s = -\frac{1}{18}\,\frac{P_2^1}{Q_2^1}\,\underline{S}_{e_{12}}^{(2)} - \frac{1}{2}\,\frac{P_1^{1'}}{Q_1^{1'}}\left[\underline{R}_{o_{11}}^{(2)} - 2\,\underline{Y}_{e_{oo}}^{(2)}\right]$$

$$\frac{2}{F^2}\cdot\underline{E}_2^s = -\frac{2}{675}\,\frac{P_3^1}{Q_3^1}\,\underline{S}_{e_{13}}^{(2)} + \frac{1}{54}\,\frac{P_2^{1'}}{Q_2^{1'}}\left[\underline{R}_{o_{12}}^{(2)} - 6\,\underline{Y}_{e_{01}}^{(2)} - \frac{3}{2}\,\underline{S}_{e_{11}}^{(2)}\right]$$

$$-\left\{\frac{6}{75} + \frac{1}{12}\,\frac{Q_1^1}{P_1^1}\,\frac{P_2^{1'}}{Q_2^{1'}} - \frac{12}{5}\,\frac{P_1^1}{Q_1^1} + \frac{Q_o(2\xi_o^2-1) - 2\xi_o}{P_1^1\,Q_2^1}\right\}\frac{P_1^1\,Q_o}{3Q_1^1}\,\underline{S}_{e_{11}}^{(2)}$$

$$-\frac{1}{10}\,\frac{P_1^1}{Q_1^1}\,\underline{U}_{e_{11}}^{(2)} - \underline{V}_{e_{00}}^{(2)}\left\{\frac{(1+Q_o)}{Q_o}\left[\frac{2}{75} + \frac{1}{36}\,\frac{Q_1^1}{P_1^1}\,\frac{P_2^{1'}}{Q_2^{1'}} + \right.\right.$$

$$\left.\left. + \frac{3Q_o(2\xi_o^2 - 1) - 6\xi_o}{P_1^1\,Q_2^{1'}}\right] + \frac{7}{5}\,\frac{P_1^1}{Q_1^1}\right\}.$$

Here, as in (a) above, the argument of all Legendre functions is ξ_o and the primes indicate differentiation. The previously undefined vector functions appearing

are given by the expressions

$$\underline{R}^{(2)}_{o_{mn}} \equiv \nabla\phi^{(2)}_{o_{mn}} \wedge \underline{r}, \ \underline{S}^{(2)}_{e_{mn}} \equiv \nabla\phi^{(2)}_{e_{mn}}$$

$$\underline{Y}^{(2)}_{e_{mn}} \equiv \nabla\phi^{(2)}_{e_{mn}} \wedge \hat{i}_y$$

$$\underline{U}^{(2)}_{e_{11}} \equiv \underline{r}\,\phi^{(2)}_{e_{11}} - 2\,r^2\ \nabla\phi^{(2)}_{e_{11}} + 6\,\hat{i}_x\,\phi^{(2)}_{e_{11}}$$

$$\underline{V}^{(2)}_{e_{oo}} \equiv \hat{i}_x\,\phi^{(2)}_{e_{oo}} - x\ \nabla\phi^{(2)}_{e_{oo}}$$

where the functions $\phi^{(2)}_{o_{mn}}$ are spheroidal harmonics of the forms given in (3.22), written specifically

$$\phi_{\substack{e \\ o}mn} = P_n^m(\eta) Q_n^m(\xi) \begin{matrix}\cos \\ \sin\end{matrix}\, m\,\phi .$$

Corresponding coefficients for the scattered magnetic field are expressible in similar fashion. For explicit forms, see Justice (1956). (Note however an inconsistency in definitions of the coefficients $\underline{E}^s_{1,2}$, $\underline{H}^s_{1,2}$ appearing there, viz. the series are written in one place in powers of (ik) and in another in powers of (ikc), c being the semi-focal length.)

The coefficients in the near-field series obtained via the potential function method are also given in the above reference. These are extremely complicated

and voluminous, however, and we list instead the results for the far field, which are easily obtained from Stevenson's formulas for the general ellipsoid by specializing to the case of a prolate spheroid (cf. Stevenson 1953, a,b).

Assume a spheroid of major axis 2a, minor axis 2b, dielectric constant and permeability ϵ and μ, respectively, immersed in a vaccum with major axis in the z axis and struck by a plane wave with harmonic time dependence $e^{-i\omega t}$, wavelength λ, and propagation, electric, and magnetic vectors specified respectively by the three sets of direction cosines ℓ, m, n; ℓ_1, m_1, n_1; ℓ_2, m_2, n_2. Without loss of generality we can set m = 0. Then the (spherical polar) components of the scattered far fields are given by the expressions

$$E_\theta = H_\phi = \left(\frac{\partial P}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \bar{P}}{\partial \phi}\right)\frac{e^{ikR}}{R}$$

$$E_\phi = -H_\theta = \left(\frac{1}{\sin\theta}\frac{\partial P}{\partial \phi} - \frac{\partial \bar{P}}{\partial \theta}\right)\frac{e^{ikR}}{R}$$

where, to order k^4,

$$P \equiv k^2 (K_1\alpha + K_2\beta + K_3\gamma) + k^4\Big[L_1\alpha + L_2\beta + L_3\gamma +$$

$$+ M_1\alpha^2 + M_2\beta^2 + M_3\gamma^2 + N_1\beta\gamma + N_2\gamma\alpha + N_3\alpha\beta$$

$$-\frac{1}{30}(K_1\alpha + K_2\beta + K_3\gamma)(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)\Big]$$

and $\bar{P}$ is obtained from this by making the substitutions

$$(\ell_1, m_1, n_1) \to (\ell_2, m_2, n_2)$$

$$(\ell_2, m_2, n_2) \to -(\ell_1, m_1, n_1)$$

$$\epsilon \leftrightarrow \mu .$$

Here α, β, γ, are direction cosines of the field point (R, θ, ϕ) and the quantities K, L, M, N are defined as follows;

$$K_1 = \frac{2}{3}(\epsilon - 1)\, f_1(\epsilon)\ell_1$$

$$K_2 = \frac{2}{3}(\epsilon - 1)\, f_2(\epsilon) m_1$$

$$K_3 = \frac{2}{3}(\epsilon - 1)\, f_3(\epsilon) n_1$$

$$15\, L_1 \equiv f_1(\epsilon)\left\{(\epsilon - 1)\ell_1\left[\frac{1}{3}(5b^2 - a^2) - (b^2\ell^2 + a^2 n^2)\right] - \epsilon\, a^2 n m_2\right\}$$

$$+ \left[f_1(\epsilon)\right]^2 \ell_1 \left\{(\epsilon - 1)\left[(\epsilon - 2) I + \epsilon b^2 I_b - \frac{4}{C}\right] + \epsilon^2 \mu \left(\frac{a^2 + b^2}{ab^2}\right)\right\}$$

$$+ f_1(\epsilon) g_1(\mu) n m_2 \left\{(I_b - I_a)\left[\frac{\mu}{2} b^2 + \frac{\mu - 2}{2} a^2\right] + \epsilon\mu\left(\frac{a^2 - b^2}{ab^2}\right)\right\}$$

$$+ \left[f_1(\epsilon)\right]^2 g_1(\mu)(I_b - I_a)\ell_1\left[(\epsilon - 1) k_1(\mu) - \epsilon\mu(\epsilon\mu - 2)\left(\frac{a^2 - b^2}{ab^2}\right)\right]$$

$$15\, L_2 \equiv f_2(\epsilon)\left\{(\epsilon - 1) m_1\left[\frac{1}{3}(5b^2 - a^2) - (a^2 n^2 + b^2 \ell^2)\right] + \epsilon(a^2 n \ell_2 - b^2 \ell n_2)\right\}$$

$$+ \left[f_2(\epsilon)\right]^2 m_1 \left\{(\epsilon - 1)\left[(\epsilon - 2) I + \epsilon b^2 I_b - \frac{4}{a}\right] + \epsilon^2 \mu \left(\frac{a^2 + b^2}{ab^2}\right)\right\} +$$

$$+ f_2(\epsilon)g_2(\mu)\left\{(I_a - I_b)\left[\frac{\mu}{2}(a^2+b^2)(n\ell_2+\ell n_2)-n\ell_2 a^2-\ell n_2 b^2\right]-\epsilon\mu\left(\frac{a^2-b^2}{ab^2}\right)(n\ell_2-\ell n_2)\right\}$$

$$+\left[f_2(\epsilon)\right]^2 g_2(\mu)(I_a - ?)m_1\left[(\epsilon-1)k_2(\mu)+\epsilon\mu(\epsilon\mu-2)\left(\frac{a^2-b^2}{ab^2}\right)\right]$$

$$15\, L_3 \equiv f_3(\epsilon)\left\{(\epsilon-1)n_1\left[\frac{2}{3}(3a^2-b^2)-(a^2n^2-b^2\ell^2)\right]+\epsilon b^2 \ell m_2\right\}$$

$$+\left[f_3(\epsilon)\right]^2 \quad (\epsilon-1)\left[(\epsilon-2)I+\epsilon a^2 I_a - \frac{4a}{b^2}\right]+\epsilon^2\mu\cdot\frac{2}{a}\ \Big\}$$

$$M_1 \equiv \frac{(\epsilon-1)}{45\,Q}\left[(\epsilon-1)(I_{bb}nn_1 - 2I_{ab}\ell\ell_1)+\frac{2\epsilon}{ab^4(2a^2+b^2)}(2b^2\ell\ell_1-a^2nn_1)\right]$$

$$M_2 \equiv \frac{(\epsilon-1)}{45\,Q}\left[(\epsilon-1)(I_{ab}\ell\ell_1+I_{bb}nn_1)-\frac{2\epsilon}{ab^4(2a^2+b^2)}(a^2nn_1+b^2\ell\ell_1)\right]$$

$$M_3 \equiv \frac{(\epsilon-1)}{45\,Q}\left[(\epsilon-1)(I_{ab}\ell\ell_1-2I_{bb}nn_1)+\frac{2\epsilon}{ab^4(2a^2+b^2)}(2a^2nn_1-b^2\ell\ell_1)\right]$$

$$15\, N_1 \equiv \frac{1}{3}(\mu-1)f_1(\mu)(a^2-b^2)\ell_2+n\,m_1\,g_1(\epsilon)\left[\frac{\epsilon}{2}(a^2+b^2)-a^2\right]$$

$$-f_1(\mu)g_1(\epsilon)\ell_2\left[\epsilon\mu\left(\frac{a^2-b^2}{ab^2}\right)+(\mu-1)k_1(\epsilon)\right]^*$$

$$15\, N_2 \equiv -\frac{1}{3}(\mu-1)f_2(\mu)(a^2-b^2)m_2+g_2(\epsilon)\left[\frac{\epsilon}{2}(a^2+b^2)(n\ell_1+\ell n_1)-a^2n\ell_1-b^2\ell n_1\right]$$

$$+f_2(\mu)\,g_2(\epsilon)m_2\left[\epsilon\mu\left(\frac{a^2-b^2}{ab^2}\right)-(\mu-1)k_2(\epsilon)\right]$$

$$15\, N_3 \equiv g_3(\epsilon)\ell m_1(\epsilon-1)b^2$$

* Note that in Stevenson's article the expression for N_1 specialized to the case of a perfectly conducting ellipsoid (Stevenson, 1953b, Sec. 6(3), p. 1148) contains a numerical error consisting of the omission of a factor of 2 in the denominator of the second term. The same error is carried over in the corresponding expression in Sec. 6(5), p. 1150.

where
$$f_1(\omega) = f_2(\omega) = \left[(\omega-1)\, I_b + \frac{2}{ab^2}\right]^{-1}$$

$$f_3(\omega) = \left[(\omega-1)\, I_a + \frac{2}{ab^2}\right]^{-1}$$

$$g_1(\omega) = g_2(\omega) = \left[(\omega-1)(a^2+b^2)\, I_{ab} + \frac{2}{ab^2}\right]^{-1}$$

$$g_3(\omega) = \left[(\omega-1) \cdot 2b^2\, I_{bb} + \frac{2}{ab^2}\right]^{-1}$$

with $\omega = \mu, \epsilon$, and

$$k_1(\epsilon) = -k_2(\epsilon) = (1-\frac{\epsilon}{2})(b^2\, I_b - a^2\, I_a) - \frac{\epsilon}{2}(a^2\, I_b - b^2\, I_a).$$

Also
$$I \equiv \frac{1}{\sqrt{a^2-b^2}} \log \frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}$$

$$I_a \equiv \frac{I-\frac{2}{a}}{2(a^2-b^2)}, \qquad I_b \equiv \frac{2\frac{a}{b^2}-I}{2(a^2-b^2)}.$$

$$I_{ab} \equiv \frac{2a^2+4b^2-3ab^2\, I}{2ab^2(a^2-b^2)^2}$$

$$I_{bb} \equiv \frac{4a^3-10\,ab^2+3b^4\, I}{8\, b^4(a^2-b^2)^2}$$

and finally

$$Q \equiv (\epsilon-1)^2\, I_{ab}(2\, I_{bb} + I_{ab}) - \frac{4\,\epsilon(\epsilon-1)}{ab^2}\left(J + \frac{a^2+2b^2}{b^2(2a^2+b^2)}\, J'\right) + \frac{4\,\epsilon^2}{a^2 b^6 (2a^2+b^2)}$$

with
$$J \equiv \frac{1}{8ab^4(a^2 - b^2)^3}\left[4a^4 - 18a^2b^2 - 16b^4 + 15\,ab^4 I\right]$$

and
$$J' \equiv \frac{1}{8b^2(a^2 - b^2)^3}\left[4a^3 + 26ab - 3b^2(4a^2 + b^2)I\right].$$

The convergence properties of the power series representation for the scalar case are discussed at length by Senior (1961) (see Fig. 15), but the conclusions reached there do not necessarily hold for vector problems. Results computed from the first term of the power series for various polarizations and incident directions have been obtained at the Radiation Laboratory (Sleator, 1959) and appear in Figure 16 . Some idea of the accuracy of the two-term approximation in certain particular cases can be obtained from Figure 17.

The expansion of the scattered electric field of a conducting spheroid, with plane wave incident nose-on, considered by Senior for the low frequency region is given in eq. (3.37). The coefficients A_n, B_n which express the field in terms of the vector wave functions $\underline{M}_{o_{1n}}$, $\underline{N}_{e_{1n}}$ are expanded in powers of $c \equiv kF$ in the forms

$$A_n = -\frac{c^{n+2}}{b_n}\sum_{r=0}^{\infty}(-ic)^r A_r^n$$

$$B_n = -i\,\frac{c^{n+2}}{b_n}\sum_{r=0}^{\infty}(-ic)^r B_r^n$$

where $$b_n \equiv 2^{2n+1} \frac{(n-\frac{1}{2})!\,(n+\frac{1}{2})!}{\pi(n+1)!} \;.$$

Explicit expressions for the coefficients A_r^n, B_r^n have been worked out for general n and for $r = 0, 1$ in the case of A_r^n and $r = 0, 1, 2$ for B_r^n.

These values are sufficient to give the first two non-vanishing terms in the power series expansion of the scattered field. Considerable excess information is contained in these forms, in that the index n can take on any value, but without a larger range of r, no more terms in the field expansion are completely known. The available expressions are given in the following tables. As in the previous table, all Legendre functions have argument ξ_0.

I. Expressions for the A_r^n

r	n even	n odd
0	0	$\left[D_n - \frac{1}{2}\delta_{1,n}\right]\frac{P_1^1}{Q_1^1}$
1	$-\frac{1}{9}\left\{\frac{2i}{\pi}D_n + \frac{1}{6}\delta_{2,n}\right\}\frac{P_2'}{Q_2'}$ $+\left\{\frac{2i}{\pi}D_n - \frac{(2n+1)(n+2)}{(2n+3)n^2}D_{n+1}\right\}\frac{P_1}{Q_1}$	0

II. Expressions for the B_r^n

r	n even	n odd
0	0	0
1	$-\frac{1}{9}\left\{\frac{2i}{\pi} D_n - \frac{1}{6}\delta_{2,n}\right\}\frac{P_2}{Q_2}$ $+\left\{\frac{2i}{\pi} D_n - \frac{(2n+1)(n+2)}{(2n+3)n^2} D_{n+1}\right\}\frac{P_1'}{Q_1'}$	0
2	0	$-\frac{1}{225}\left[D_n - \frac{3}{2}\delta_{1,n} + \frac{1}{6}\delta_{3,n}\right]\frac{P_3}{Q_3} +$ $+\frac{1}{36}\left\{D_n + \frac{3}{2}\delta_{1,n} \; \frac{8i}{\pi}\frac{(2n+1)(n+2)}{(2n+3)n^2} D_{n+1}\right\}\frac{P_2'}{Q_2'} +$ $+\left\{\left(\frac{31}{100} - \frac{8n^2+14n+9}{2(2n-1)^2(2n+3)^2}\right) D_n + \frac{27}{100}\delta_{1,n} -\right.$ $\left. - \frac{21}{\pi}\frac{(2n+1)(n+2)}{(2n+3)n^2} D_{n+1}\right\}\frac{P_1}{Q_1} -$ $-\frac{1}{5}\left\{D_n - \frac{1}{2}\delta_{1,n}\right\}\frac{P_1 Q_{-1}}{(Q_1)^2}$

Here $\delta_{m,n}$ is the Kronecker delta and

$$D_n = 2^n \cdot \frac{(2n+1)}{n(n+1)} \cdot \frac{\left(\frac{n-1}{2}\right)!\,\left(\frac{n+1}{2}\right)!}{(n+1)!} \cdot i^{n-1} .$$

5. Variational Forms (See Sec. 3.1.1 p. 85, also Sleator, 1960)

The variational coefficients $C_{\mu\nu}$, B_ν defined in eqs. (3.47), (3.48) are written more explicitly as follows: For $\mu \leqslant \nu$, $\mu+\nu$ even,

$$C_{\mu\nu} = -8\pi^2 F^4 \xi_o^2 (\xi_o^2 - 1)^{3/2} k^3 i^{\mu+3\nu+1} \int_0^\pi \frac{\Gamma_\mu(\xi_o, k, \psi)\Delta_\nu(\xi_o, k, \psi)}{(\xi_o^2 - \cos^2\psi)^{5/2}} \sin\psi \, d\psi$$

where

$$\Gamma_\mu(\xi_o, k, \psi) \equiv \frac{\mu}{\rho}\left[\cos\psi P_{\mu-1}(\cos\psi) - \xi_o^2 P_\mu(\cos\psi)\right] h_\mu^{(1)}(\rho) + (\xi_o^2 - \cos^2\psi) P_\mu(\cos\psi) h_{\mu+1}^{(1)}(\rho)$$

$$\Delta_\nu(\xi_o, k, \psi) \equiv \frac{\nu}{\rho}\left[\cos\psi P_{\nu-1}(\cos\psi) - \xi_o^2 P_\nu(\cos\psi)\right] j_\nu(\rho) + (\xi_o^2 - \cos^2\psi) P_\nu(\cos\psi) j_{\nu+1}(\rho)$$

and $\rho = \dfrac{kF\, \xi_o \sqrt{\xi_o^2 - 1}}{(\xi_o^2 - \cos^2\psi)^{1/2}}$, ξ_o being the coordinate of the scattering surface.

For $\nu < \mu$, the subscripts on Γ and Δ should be interchanged, and for $\mu + \nu$ odd the integral vanishes. Further,

$$B_\nu = 4\pi F^2 (\xi_o^2 - 1) i^\nu k \frac{d j_\nu(ka)}{d(ka)} .$$

The stationary value J_o of the variational quantity J defined in eq. (3.43) can be written

$$J_o = 4\pi \left(\sum_\mu A_\mu B_\mu\right)^{-1}$$

where the quantities A_μ are the solution of the linear system (3.46). These have been computed for a particular spheroid (a/b = 10) at a particular frequency

(ka = 1.40) for the range $\mu = 0 - 4$ inclusive. The values are shown in the following table:

μ	Re A_μ	Im A_μ
0	$6.92536 \cdot 10^{-1}$	$-5.44415 \cdot 10^{-3}$
1	$-1.69158 \cdot 10^{-3}$	1.15989
2	$-5.82917 \cdot 10^{-1}$	$9.76788 \cdot 10^{-4}$
3	$1.31725 \cdot 10^{-4}$	$-1.70612 \cdot 10^{-1}$
4	$3.52111 \cdot 10^{-2}$	$-3.00564 \cdot 10^{-5}$

The resulting potential distribution over the surface of the hard spheroid struck by a plane wave nose-on is plotted in Figs. 1 - 18. The normalized backscattering cross section $\sigma = 4 \frac{a^2}{b^4} \left| J_o \right|^{-2}$ computed for this case has a value 1.105, as compared to the value 1.091 given by the ordinary wave-function series, (see Fig. 9).

6. Geometric and Physical Optics (See Sec. 3.1.2.1 p. 89)

The geometric optics cross section of a spheroid with transmitter on the axis of symmetry and receiver separated from this axis by an angle $\beta < \pi$ is given in eq. (3.49) and plotted in Fig. 19. By the theorem quoted in this context, the monostatic cross section is thus also given for values of the polar angle $\theta = \beta/2$.

The physical optics integral is given in (3.53) but cannot be evaluated exactly except in certain special cases. Some numerical evaluations of cross section have

been carried out (Siegel et al, 1955a) and are shown in Fig. 19. The bistatic cross section at those angles where exact evaluation of the physical optics integral is possible (see p. 95) is given in Fig. 20.

The total scalar scattering coefficient (cf. Jones 1957) of a prolate spheroid with plane wave incident nose-on, as obtained via the physical optics approximations, is given in eq. (3.58) as

$$\sigma \simeq 2 + 2\, b_o \left(\frac{a}{kb^2}\right)^{2/3}, \quad kb^2/a >> 1.$$

The values of the coefficient b_o are:

Hard spheroid (Neumann boundary condition): $b_N = -.8640$

Soft spheroid (Dirichlet boundary condition): $b_D = .9962$.

For broadside incidence, the total scattering coefficient can be written

$$\sigma \simeq 2 + 2b_o (kb)^{-2/3} \cdot C$$

where b_o is as given above and the correction factor C depends on the axis ratio as illustrated by the following table of values:

b/a =	1.0	.8	.6	.4	.2
C =	1	.874	.761	.673	.608

For the electromagnetic problem with nose-on incidence, the total cross section is given by eq. (3.62), viz.

$$\sigma_T \simeq 2 + (b_D + b_N)\left(\frac{a}{kb^2}\right)^{2/3}$$

with b_D and b_N as above. For broadside incidence there are two distinct results, one for parallel polarization ($\underline{E}^i || a$) and the other for perpendicular ($\underline{E}^i \perp a$). These can be written

$$\sigma_{||,\perp} \simeq 2 + (b_D + b_N)(kb)^{-2/3} \cdot C_{||,\perp}$$

where the correction factors $C_{||}$, $C_{\perp}$ are given for various axis ratios as follows:

b/a =	1.0	.9	.8	.6	.4	.2
$C_{\|\|}$ =	1	2.1	3.09	5.08	6.68	8.11
$C_{\perp}$ =	1	-0.21	-1.41	-3.66	-5.47	-6.93

7. Modified Geometrical Theory. (See Sec. 3.1.2.2 p. 100, also Levy and Keller, 1959).

The scalar diffracted field at the point $P(\xi, \eta)$ produced by a soft prolate spheroid ξ_o with point source on the axis of symmetry at the point $Q(\xi_1, 1)$ is given by the expression

$$u_d(P) = \frac{\sqrt{\pi}\; e^{i\frac{\pi}{12}} \left(\xi_o\sqrt{\xi_o^2 - 1}\right)^{1/6}}{2^{3/2}\, 6^{4/3}\, k^{1/6}\, F^{7/6} \left[(\xi_1^2 - \xi_o^2)(\xi_1^2 - 1)(1-\eta^2)(\xi^2 - 1)\right]^{1/4}}$$

$$\cdot \sum_{n=0}^{\infty} \frac{f_n(\eta_2) - i\, f_n(\eta_3)}{\left[A'(q_n^{(1)})\right]^2 \left[1 + \exp\left\{-2ikF\left(\int_{-1}^{1}\sqrt{\frac{\xi_o^2 - \eta^2}{1-\eta^2}}\, d\eta - \tau_n^{(1)} \left(\frac{\xi_o\sqrt{\xi_o^2 - 1}}{kF}\right)^{2/3} \int_{-1}^{1} \frac{d\eta}{\sqrt{(\xi_o^2 - 1)(1-\eta^2)}}\right)\right\}\right]}$$

Here η_2 and η_3 are coordinates of the points of tangency P_2, P_1 of rays through P, (See Fig. 4). The function f_n is defined as

$$f_n(\eta_j) \equiv \left[\frac{\xi_o^2 - \eta_j^2}{(\xi\eta - \xi_o\eta_j)^2 + \left(\sqrt{(\xi^2-1)(1-\eta^2)} \pm \sqrt{(\xi_o^2-1)(1-\eta_j^2)}\right)^2} \right]^{1/4} \cdot \exp\Bigg\{ ikF$$

$$\cdot \left[\frac{\sqrt{(\xi_1^2-1)(\xi_1^2-\xi_o^2)}}{\xi_1} + \left((\xi\eta - \xi_o\eta_j)^2 + \left(\sqrt{(\xi^2-1)(1-\eta^2)} \pm \sqrt{(\xi_o^2-1)(1-\eta_j^2)}\right)^2 \right)^{1/2} \right.$$

$$\left. - \int_{\eta_1}^{\eta_j} \sqrt{\frac{\xi_o^2-\eta^2}{1-\eta^2}}\, d\eta - \tau_n^{(1)} \left(\frac{\xi_o\sqrt{\xi_o^2-1}}{kF} \right)^{2/3} \int_{\eta_1}^{\eta_j} \frac{d\eta}{\sqrt{(\xi_o^2-\eta^2)(1-\eta^2)}} \right] \Bigg\}$$

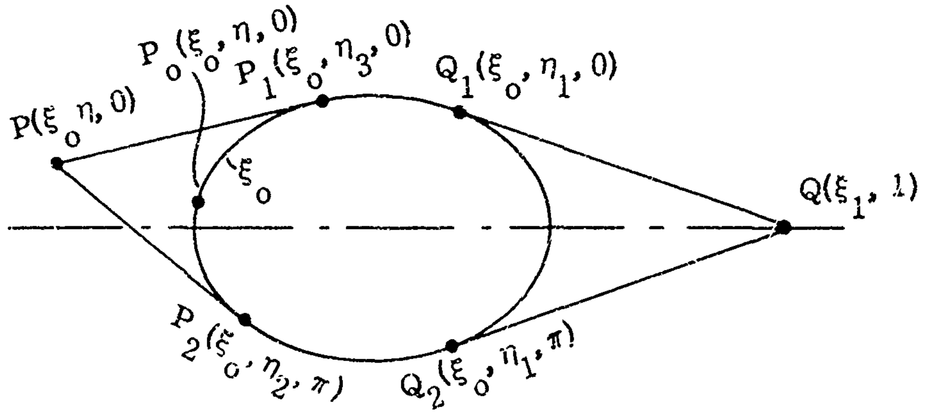


FIG. 4

where j = 2, 3 and the ambiguous signs are fixed as follows:

- if j = 2 <u>or</u> $\xi\eta + \xi_o > 0$

+ if j = 3 <u>and</u> $\xi\eta + \xi_o < 0$.

Also $\tau_n^{(1)} \equiv 6^{-1/3} e^{i\pi/3} q_n^{(1)}$ where $q_n^{(1)}$ is the n^{th} zero of the Airy function

$A(t) = \int_0^\infty \cos(z^3 - tz)\, dz$, i.e. $A(q_n^{(1)}) = 0$ for all n, and in the expression above for

$u_d(P)$, $A'(q_n^{(1)})$ is the derivative of this function evaluated at $q_n^{(i)}$. The corresponding result for the hard spheroid is obtained from the above by replacing $\tau_n^{(1)}$ by $\tau_n^{(2)}$ and $\left[A'(q_n^{(1)})\right]^{-2}$ by $3\left\{q_n^{(2)}\left[A(q_n^{(2)})\right]^2\right\}^{-1}$, where $\tau_n^{(2)} = \dfrac{e^{i\pi/3}\, q_n^{(2)}}{6^{1/3}}$ and $q_n^{(2)}$ is defined by the relation $A'(q_n^{(2)}) = 0$ for all n.

On the surface of the spheroid, which is a caustic of the diffracted rays, these expressions must be modified (see p. 107). The corrected expression for the field on the surface of the hard spheroid, specialized for plane wave incidence, is

$$u_d(P_o) = \frac{\pi \xi_o^{1/4}}{\left[(\xi_o^2 - \eta^2)(1-\eta^2)\right]^{1/4}} \sum_{n=1}^{\infty} \frac{\exp G_n(0,\ \eta) - i\exp\left[G_n(\eta, -1) + G_n(-1, 0)\right]}{(1 + \exp 2G_n(-1, 1))\, q_n^{(2)} A(q_n^{(2)})},$$

where

$$G_n(\alpha, \beta) \equiv -ikF\left[\int_\alpha^\beta \sqrt{\frac{\xi_o^2 - \eta^2}{1-\eta^2}}\, d\eta + \tau_n^{(2)}\left(\frac{\xi_o\sqrt{\xi_o^2 - 1}}{kF}\right)^{2/3} \int_\alpha^\beta \frac{d\eta}{\sqrt{(\xi_o^2 - \eta^2)(1-\eta^2)}}\right].$$

At a point on the axis at large distance z from the scatterer in the direction of the source (i.e. backscattering direction) the leading term of the series for the geometrical (reflected) field is

$$u_g = \pm \frac{b^2}{2\,az} e^{ik(z - 2a)}$$

where a and b are, as usual, the major and minor semi-axes and the positive sign holds for the hard spheroid, negative sign for the soft. The leading term of

the total backscattered field $u = u_g + u_d$ is finally

$$u = \frac{b^2}{2az}\, e^{ik(z-2a)} \left\{ \pm 1 - \frac{2\,\delta\,\pi^2 (kF)^{1/3}\, \xi_o^{5/3}\, e^{i\Gamma}}{q_1^{(2)} A^2(q_1^{(2)})\, 6^{1/3} (\xi_o^2 - 1)^{2/3}} \right\}$$

where the signs are as before, the quantity δ is defined as

$$\delta = \begin{cases} 1 & \text{for the hard spheroid} \\ \dfrac{q_1^{(2)} A^2(q_1^{(2)})}{3\left[A'(q_1^{(1)})\right]^2} & \text{for the soft,} \end{cases}$$

and

$$\Gamma \equiv \frac{2\pi}{3} + 4ka - 2\tau_1^{(j)} (kF)^{1/3} \left(\xi_o \sqrt{\xi_o^2 - 1}\right)^{2/3} \int_0^1 \frac{d\eta}{(\xi_o^2 - \eta^2)(1-\eta^2)} - 2kF \int_0^1 \sqrt{\frac{\xi_o^2 - \eta^2}{1-\eta^2}}\, d\eta$$

with $j = 1(2)$ for the soft (hard) case, and $\tau_1^{(j)}$ as defined above. Numerical values of the constants are given by Levy and Keller as

$$q_1^{(1)} = 3.372134 \qquad q_1^{(2)} = 1.469354$$

$$A'(q_1^{(1)}) = -1.059053 \qquad A(q_1^{(2)}) = 1.16680$$

Accuracy of this approximation has not been determined in general. It has been shown (cf. Kazarinoff and Ritt, 1959) that it is applicable only when the wavelength is small relative to the radius of curvature at the tip of the spheroid.

8. Asymptotic Solutions (see Sec. 3.1.2.3, p. 112)

a. Fat spheroid (cf. Kazarinoff and Ritt, 1959 a)

The scalar field in the shadow region on the surface of a hard spheroid struck by a plane wave nose-on is given by the series

$$u(\xi_o, \eta) = \sum_r R_r$$

$$\text{where } R_r = \begin{cases} A_r\left\{\dfrac{e^{i\nu_r d(\eta) + i\frac{\pi}{4}} + e^{i\nu_r d^*(\eta) - i\frac{\pi}{4}}}{1 + e^{i\nu_r L}}\right\} & \text{in the vicinity of the shadow boundary} \\ B_r\left\{\dfrac{e^{i\nu_r L/4}}{1 + e^{i\nu_r L}}\right\} & \text{in the vicinity of the tip.} \end{cases}$$

Here

$$A_r = \frac{2}{\sqrt{3}}\, i\left\{h_r H^{(2)}_{\frac{1}{3}}(h_r)\left[(1-\eta^2)(1-\epsilon^2\eta^2)\right]^{1/4}\right\}^{-1}$$

$$B_r = \sqrt{\frac{2\pi}{3}}\;\frac{i\left[d(\eta) - \frac{L}{4}\right]^{1/2} J_o\left[\nu_r\left(d^*(\eta) - \frac{L}{4}\right)\right]}{b^{3/2}\xi\, h_r H^{(2)}_{\frac{1}{3}}(h_r)\left[(1-\eta^2)(1-\epsilon^2\eta^2)\right]^{1/4}}$$

$$d(\eta) = b\left[S(-\eta) - S(0)\right]$$

$$d^*(\eta) = -b\left[S(-\eta) + S(0)\right]$$

$$S(\eta) = \frac{-1}{\sqrt{\xi_r^2 - 1}} \int_{\eta}^{1} \sqrt{\frac{\xi_r^2 - t^2}{1 - t^2}}\, dt$$

$L = -4b\, S(0) =$ circumference of generating ellipse

$$\nu_r = -\frac{1}{b}\left[(\xi_r^2 - 1)\gamma^2\right]^{1/2}$$

$\gamma =$ complex propagation const. $= \zeta - is$

$$\xi_r = \xi_o + e^{-i\frac{\pi}{3}}\left(\frac{\xi_o^2 - 1}{2\xi_o}\right)^{\frac{1}{3}}\left(\frac{3h_r}{2\gamma}\right)^{\frac{2}{3}}\left[1 + \frac{e^{-i\frac{\pi}{3}}(7+\xi_o^2)}{10(2\xi_o)^{4/3}(\xi_o^2-1)^{2/3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}\left(\frac{h_r}{\gamma}\right)^{\frac{2}{3}} + O\left(\frac{1}{\gamma}\right)\right]$$

$$h_r = \text{rth zero of } \frac{d}{dt}\left[t^{1/3} H_{\frac{1}{3}}^{(2)}(t)\right]$$

$2b =$ minor axis and $\epsilon =$ eccentricity $= \frac{1}{\xi_o}$.

In the limit of zero eccentricity, the result for the field near the tip is

$$R_r \xrightarrow[\epsilon \to 0]{} \frac{\sqrt{\frac{2\pi}{3}}\; i\left[\sin^{-1}(-\eta) - \frac{\pi}{2}\right]^{\frac{1}{2}} J_o\left[\gamma\sqrt{\xi_r^2 - 1}\left(\sin^{-1}(-\eta) - \frac{\pi}{2}\right)\right]}{a\zeta\, h_r\, H_{\frac{1}{3}}^{(2)}(h_r)(1-\eta^2)^{1/4}\cos\left[2\gamma\sqrt{\xi_r^2 - 1}\left(\sin^{-1}(-\eta) - \frac{\pi}{2}\right)\right]}.$$

These results have significant accuracy only on condition that

$\zeta b^2/a >> 1$, (cf. Kazarinoff and Ritt, 1959 b).

b. Thin spheroid (cf. Goodrich and Kazarinoff, 1963)

For a thin spheroid ($\kappa R_o = k\frac{b^2}{a} << 1$) at high frequencies ($ka >> 1$) with point source on the axis of symmetry at the point $(\xi_1, 1)$ the surface fields in the

shadow region are given approximately by the following expressions:

Near the tip, where $\left| c(1+\eta) \right| \ll 1$:

For the Dirichlet problem

$$\frac{\partial u(\xi_o,\eta)}{\partial \xi} \simeq -\sum_{n=0}^{N}\sum_{m=0}^{\infty} \frac{i^n n!\, e^{i\epsilon(\xi_1+1)} a^2 \cos\left[c(1+\eta)\right]}{2^{2n-1/2} c^n b^2 \left[(1-\eta^2)(\xi_1^2-1)\right]^{1/2}} R_1 X_m^{(1)}$$

For the Neumann problem

$$u(\xi_o,\eta) \simeq \sum_{n=0}^{N}\sum_{m=0}^{\infty} \cdot \frac{i^n n!\, e^{-ic(\xi_1+1)} \cos\left[c(1+\eta)\right]}{2^{2n+1/2} c^n \left[(1-\eta^2)(\xi_1^2-1)\right]^{1/2}} X_m^{(2)}$$

Near the shadow boundary, where $\left| c(1+\eta) \right| \gg 1$:

For the Dirichlet problem

$$\frac{\partial u(\xi_o\eta)}{\partial \xi} \simeq \sum_{n=0}^{N}\sum_{m=0}^{\infty} \frac{2(-1)^{n+1} a^2 e^{-ic(\xi_1+1)}}{b^2\left[(1-\eta^2)(\xi_1^2-1)\right]^{1/2}} R_1 X_m^{(1)} \cdot$$

$$\left[e^{ic(1+\eta)}\left(\frac{1+\eta}{1-\eta}\right)^{\eta+1/2} + \frac{(-1)^n i(n!)^2}{2^{4n} c^{2n+1}}\left(\frac{1+\eta}{1-\eta}\right)^{\eta+1/2} e^{-ic(1+\eta)}\right]$$

For the Neumann problem

$$u(\xi_o,\eta) \simeq \sum_{n=0}^{N}\sum_{m=0}^{\infty} \frac{(-1)^{n+1} e^{-ic(\xi_1+1)}}{\left[(1-\eta^2)(\xi_1^2-1)\right]^{1/2}} X_m^{(2)} \cdot$$

$$\left[e^{ic(1+\eta)}\left(\frac{1+\eta}{1-\eta}\right)^{\eta+1/2} + \frac{i(-1)^n (n!)^2}{2^{4n} c^{2n+1}} R_2\, e^{-ic(1+\eta)}\left(\frac{1-\eta}{1+\eta}\right)^{n+1/2}\right]$$

where $c = kF$

$$R_1 = \left[\log \frac{cb^2}{a^2}\right]^{-1}, \quad R_2 = \frac{icb^2}{2a^2}$$

$$X_m^{(j)} = e^{-4im}\left[4ic\right]^{-2m(2n+1)}(n!)^{4m} R_j^{2m}, \quad j = 1, 2 \quad .$$

These results should hold when $\frac{a}{b} = 10^n$ $(n = 2, 3, \ldots)$ and $10 < ka < 10^{2n-1}$.

9. Uniform Field Result, Thin Spheroid (see Sec, 3.2.1, p. 117, also Page and Adams,1938).

For a thin conducting spheroid in a time-harmonic, instantaneously uniform electric field parallel to the major axis and given by the expression

$$\underline{E}^i = \hat{i}_z E_o e^{-i\omega t}$$

in a medium of permittivity ϵ, permeability μ, the components of the scattered field at the point (ξ, η) in the far zone can be written approximately as

$$E_\eta \simeq \frac{2}{3} \cdot \frac{E_o \xi_o \ell c^2 s_1}{\sqrt{\xi^2 - \eta^2}} \frac{\left[u_1(\eta) - \frac{2b_1 c^4}{3^2 \cdot 5^4 \cdot 7} u_3(\eta)\right]}{\left[\left(\frac{4}{9} \cdot \ell c^3 a_1\right)^2 + (b_1 m_1)^2\right]^{1/2}} e^{i(c\xi - \omega t + \frac{\pi}{2} - \gamma)}$$

$$H_\phi \simeq -\frac{2}{3}\sqrt{\frac{\epsilon}{\mu}} \frac{\xi_o E_o \ell c^2 s_1 \left[u_1(\eta) - \frac{2b_1 c^4}{3^2 \cdot 5^4 \cdot 7} u_3(\eta)\right]}{\sqrt{\xi^2 - 1}\left[\left(\frac{4}{9}\ell c^3 a_1\right)^2 + (b_1 m_1)^2\right]^{1/2}} e^{i(c\xi - \omega t + \frac{\pi}{2} - \gamma)}$$

in which

$$\ell = \left[\log \frac{\xi_o + 1}{\xi_o - 1} - 2\right]^{-1}$$

$$\gamma = \tan^{-1}\left(\frac{9\, b_i m_1}{4 \ell c^3 a_i}\right)$$

$$c = kF$$

and the remaining quantities are written as power series in c which begin as follows

$$a_1 = 1 - \frac{1}{2 \cdot 5^2} c^2 + \frac{187}{2^3 \cdot 5^4 \cdot 7^2} c^4 - \frac{26,021}{2^4 \cdot 3^4 \cdot 5^6 \cdot 7^2} c^6 + \cdots$$

$$b_1 = 1 - \frac{19}{2 \cdot 5^2} c^2 - \frac{2609}{2^3 \cdot 5^4 \cdot 7^2} c^4 + \frac{32,593}{2^4 \cdot 3^4 \cdot 5^5 \cdot 7^2} c^6 + \cdots$$

$$m_1 = 1 - \frac{1}{5} c^2 \ell + \frac{9}{5^3 \cdot 7} c^4 \ell - \frac{886}{3^4 \cdot 5^5 \cdot 7} c^6 \ell + \cdots$$

$$s_1 = 1 - \frac{2}{5^4 \cdot 7} c^4 + \frac{8}{3 \cdot 5^6 \cdot 7} c^6 + \cdots$$

$$u_1 = P_1^1(\xi) - \frac{1}{3 \cdot 5^2} P_3^1(\xi)\, c^2 + \left[\frac{2}{3^2 \cdot 5^4} P_3^1(\xi) + \frac{1}{3^2 \cdot 5^2 \cdot 7^2} P_5^1(\xi)\right] c^4$$

$$- \left[\frac{31}{3 \cdot 5^5 \cdot 7^2 \cdot 11} P_3^1(\xi) + \frac{4}{3 \cdot 5^4 \cdot 7^2 \cdot 13} P_5^1(\xi) + \frac{1}{3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13} P_7^1(\xi)\right] c^6 + \cdots$$

$$u_3 = P_3^1(\xi) + \left[\frac{6}{5^2 \cdot 7} P_1^1(\xi) - \frac{2}{3^3 \cdot 7} P_5^1(\xi)\right] c^2 - \left[\frac{4}{5^4 \cdot 7} P_1^1(\xi) - \frac{4}{3^6 \cdot 5 \quad 7 \cdot 13} P_5^1(\xi)\right.$$

$$\left. - \frac{5}{2^3 \cdot 7 \cdot 11^2 \cdot 13} P_7^1(\xi)\right] c^4 + \cdots$$

Accuracy of these forms has not been established. They should apply reasonably well for the case of an incident plane wave with electric vector parallel to the major axis provided the wavelength is large compared to the minor axis.

10. Traveling Wave Formula (see Sec. 3.2.1 p. 119, also Siegel, 1959)

The backscattering cross section of a long, thin, conducting body struck by a plane electromagnetic wave with propagation vector $\underline{P}$ making an angle θ with the major axis, and electric vector in the plane of $\underline{P}$ and the axis can be written

$$\sigma = \frac{\gamma^2 \lambda^2}{\pi Q^2} \left[f(\theta) \right]^{4*}$$

where

$$f(\theta) = \frac{\sin\theta}{1 - p\cos\theta} \sin\left[\frac{kL}{2p}(1 - p\cos\theta)\right]$$

and

$$Q = -\frac{2}{p^2} + \frac{\mathrm{Cin}\left[\frac{kL}{p}(1+p)\right] - \mathrm{Cin}\left[\frac{kL}{p}(1-p)\right]}{p^3} +$$

$$+ \frac{1}{2p^3}\left\{(p-1)\cos\left[\frac{kL}{p}(1+p)\right] + (p+1)\cos\left[\frac{kL}{p}(1-p)\right] + (p^2-1)\frac{kL}{p}\left(\mathrm{Si}\left[\frac{kL}{p}(1+p)\right] - \mathrm{Si}\left[\frac{kL}{p}(1-p)\right]\right)\right\}.$$

Here Si is the sine integral

Cin is the modified cosine integral

γ = voltage reflection coefficient

---

*This formula is in error in Siegel (1959) and Crispin et al (1959).

p = relative phase velocity

L = length of body

λ = wavelength .

The relative phase velocity p is determined by the actual path length along the surface relative to the distance in the axial direction. The voltage reflection coefficient depends largely on the angle θ and on the shape of the body at the tips and must be determined by analogy or experiment. The values used by Siegel for the 10:1 prolate spheroid in three distinct ranges of θ are as follows

θ = 0-40^o	40-60^o	60-75^o
γ = .33	.7	1.0

The theory breaks down at $\theta = 0$ and in the region about $\pi/2$. Comparison with experiment is illustrated for the 10:1 spheroid in Fig. 26, p. 203.

11. Perturbation of Sphere Solution (see Sec. 3.2.2, p. 119)

a. Vector Case (cf. Mushiake 1956)

The normalized backscattering cross section of a fat spheroid specified in spherical polar coordinates by the expression

$$r = a(1 - \nu \sin^2 \theta)^{-1/2} \text{ with } \nu = 1 - a^2/b^2, |\nu| << 1$$

and struck by a plane wave whose propagation direction makes an angle α with the axis of symmetry can be written for θ polarization ($\underline{E}^i \|$ plane of incident direction and axis) and ϕ polarization ($\underline{E}^i \perp$ said plane), respectively, as

$$\frac{\sigma_\theta(\alpha)}{\pi a^2} = \left|\eta_\theta\right|^2, \quad \frac{\sigma_\phi(\alpha)}{\pi a^2} = \left|\eta_\phi\right|^2$$

where, to first order in ν,

$$\eta_{\theta,\phi} \simeq \frac{4}{ka} \sum_{n=1}^{\infty} (-1)^{n+1}(2n+1)\, \alpha_n(ka)\,\delta_n(ka) + \nu\, \eta'_{\theta,\phi}$$

with

$$\alpha_n(ka) = \frac{ka}{2} \left\{ j_n'(ka) - \frac{j_n(ka)}{h_n^{(1)}(ka)} \left[h_n^{(1)}(ka) \right]' \right\}$$

$$\delta_n(ka) = \frac{i}{2\left[ka\, h_n^{(1)}(ka)\right]'}$$

and

$$\eta'_\theta = 2 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \alpha_n \frac{m P_n^m(\cos\alpha)}{\sin\alpha} \sum_{r=m}^{\infty} i^r \left[\alpha_r I_{nr}^{1m} \frac{m P_r^m(\cos\alpha)}{\sin} - \right.$$

$$\left. - im\, \delta_r I_{nr}^{2m} \frac{d P_r^m(\cos\alpha)}{d\alpha} \right] + i^{n+1} \delta_n \frac{d P_n^m(\cos\alpha)}{d\alpha} \sum_{r=m}^{\infty} i^r \left[- m\,\alpha_r I_{nr}^{2m} \right.$$

$$\left. \cdot \frac{m P_r^m(\cos\alpha)}{\sin\alpha} + i(\beta_r I_{nr}^{1m} + \gamma_r I_{nr}^{4m}) \frac{d P_n^m(\cos\alpha)}{d\alpha} \right].$$

The corresponding expression for η'_ϕ is obtained from this by making the substitutions

$$\frac{m\, P_s^m(\cos\alpha)}{\sin\alpha} \longleftrightarrow \frac{d\, P_s^m(\cos\alpha)}{d\alpha}, \quad s = n,\ r,$$

and the quantities β_r, γ_r are defined by the expressions

$$\beta_r(ka) = \frac{ka}{2}\left\{\left[\frac{\left[ka\, j_r(ka)\right]'}{ka}\right]' - \frac{\left[ka\, j_r(ka)\right]'}{\left[ka\, h_r^{(1)}(ka)\right]'} \cdot \left[\frac{\left[ka\, h_r^{(1)}(ka)\right]'}{ka}\right]'\right\}$$

$$\gamma_r(ka) = \frac{r(r+1)}{2\, ka} \cdot \left\{ j_r(ka) - \frac{\left[ka\, j_r(ka)\right]'}{\left[ka\, h_r^{(1)}(ka)\right]'}\, h_r^{(1)}(ka)\right\}.$$

The remaining quantities I_{nr}^{sm}, $s = 1, 2, 4$, are essentially definite integrals of products of Legendre functions defined as follows:

$$I_{nr}^{1m} = \Gamma_{mnr} \int_0^{\pi} \left(\frac{dP_n^m}{d\theta}\,\frac{dP_r^m}{d\theta} + m^2\, \frac{P_n^m\, P_r^m}{\sin^2\theta}\right) \sin^3\theta \; d\theta$$

$$I_{nr}^{2m} = \Gamma_{mnr} \int_0^{\pi} \left(\frac{dP_n^m}{d\theta}\, P_r^m + \frac{dP_r^m}{d\theta}\, P_n^m\right) \sin^2\theta \; d\theta$$

$$I_{nr}^{4m} = \Gamma_{mnr} \int_0^{\pi} \frac{dP_n^m}{d\theta}\, P_r^m \, \sin 2\theta \, \sin\theta \; d\theta$$

with

$$\Gamma_{mnr} = (2 - \delta_{o,m})\, \frac{(2n+1)(n-m)!\,(2r+1)(r-m)!}{n(n+1)(n+m)!\; r(r+1)(r+m)!}$$

where $\delta_{o,m}$ is the Kronecker delta.

Results computed from these formulas for various values of ν in the neighborhood of unity are shown in Figs. 21, 22. A comparison with experimental data for a particular spheroid at a particular wavelength is shown in Fig. 34.

b. Scalar case, Neumann problem with symmetric incidence and arbitrary observation direction (cf. Sleator and Ullman 1959).

If the spheroid is specified as in the vector case above and the scattered field $\phi^s(r, \theta)$ is expanded in spherical scalar wave functions

$$\phi^s(r,\theta) = \sum_{n=0}^{\infty} A_n h_n^{(1)}(kr) P_n(\cos\theta)$$

then the coefficints A_n can be written

$$A_n = i^{-n}(2n+1)\left[-\frac{\left[j_n(ka)\right]'}{\left[h_n^{(1)}(ka)\right]'} + \nu\, a_n\right]$$

with
$$a_n = \frac{i}{(2n+1)(ka)^3\left[h_n^{(1)}(ka)\right]'}\left\{\frac{(2n+1)\left[(ka)^2(n^2+n-1)-n^2(n+1)^2\right]}{\left[h_n^{(1)}(ka)\right]'(2n-1)(2n+3)}\right.$$
$$\left.+\frac{n(n-1)\left[(ka)^2-(n-2)(n+1)\right]}{2\left[h_{n-2}^{(1)}(ka)\right]'(2n-1)}+\frac{(n+1)(n+2)\left[(ka^2)-n(n+3)\right]}{2\left[h_{n+2}^{(1)}(ka)\right]'(2n+3)}\right\}.$$

The backscattering cross section is given by the expression

$$\sigma = \frac{4\pi}{k^2} \left| \sum_{n=0}^{\infty} i^{-n} A_n \right|^2 .$$

No results have been computed from these formulas as yet. Accuracy should be comparable to the vector case.

12. Weak Scatterers (see Sec. 3.3, p. 124)

a. Scalar Case

The differential scattering cross section of a thin homogeneous spheroid of interior propagation constant k_1 immersed in a medium of propagation constant k_o and struck by a plane wave propagating in the plane $\phi = 0$ at incident angle α with the major axis (θ 0) is written approximately (see Montroll and Hart, 1951) as

$$\sigma(\theta, \phi) \simeq \frac{2\pi (k_1^2 - k_o^2) m a^4 b^2}{(1+m)^2 \left[1 + \mathcal{K}^4 + 2\mathcal{K}^2 \cos 4a k_1^* \right]} \cdot \left| \omega^{-\frac{3}{2}} J_{\frac{3}{2}}(\omega) - i\mathcal{K} e^{2iak_1^*} v^{-\frac{3}{2}} J_{\frac{3}{2}}(v) \right|^2$$

where

$$\mathcal{K} = \frac{k_1 - k_o}{k_1 + k_o}$$

$$k_1^* = \sqrt{k_1^2 - k_o^2 \cos^2 \alpha}$$

$$m = \frac{\sqrt{k_1^2 - k_o^2 \cos^2 \alpha}}{k_o \sin \alpha}$$

$$\omega = a^2 k_1^{*2} + a^2 k_o^2 \sin^2 \phi + k_o^2 b^2 (\cos \alpha - \cos \phi)^2 - 2a^2 k_1^* k_o \sin \phi \cos \theta$$

$$v = a^2 k_1^{*2} + a^2 k_o^2 \sin^2 \phi + k_o^2 b^2 (\cos \alpha - \cos \phi)^2 + 2a^2 k_1^* k_o \sin \phi \cos \theta .$$

No quantitative data are available on accuracy of this result. Qualitative remarks are made in the preceding text.

An approximate result which derives from the Schrödinger equation, under the assumptions that a) the energy of the potential is small compared to that of the incident wave, and b) its range is small compared to the wavelength, is given by Greenberg (1960). If a plane scalar wave strikes a square-well potential of prolate spheroidal form, represented by the expression

$$U = -U_o(1 + i\,\delta) \qquad \text{inside}$$

$$U = 0 \qquad \text{outside}$$

at an angle α with respect to the axis of symmetry, the total scattering cross section is given approximately by the formula

$$\sigma_T = 4\pi b\,(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) \cdot \mathrm{Re}\left\{\frac{1}{2} + \frac{ib}{C\mu} \exp(iC\frac{\mu}{b}) + \right.$$

$$\left. + (\frac{b}{C\mu})^2 \left[1 - \exp(iC\frac{\mu}{b})\right]\right\}$$

where a, b are major, minor semi-axes,

$$C = ab\left[a^2 \sin^2 \alpha + b^2 \cos^2 \alpha\right]^{1/2}$$

and $\mu = \frac{U_o b}{k}(1 + i\,\delta)$,

$$\text{i.e.} \quad \underline{E}(\underline{r}) \simeq \underline{E}_o(\underline{r}) = e^{ik\hat{i}\cdot\underline{r}}\,\underline{E}^i$$

where ϵ, ϵ_o are respectively the interior and exterior dielectric constants, $\hat{i}$ is the unit vector in the incident direction, and $\underline{E}^i$ is the incident field amplitude, the quantities $J_{\perp\rightarrow\parallel}$ and $J_{\parallel\rightarrow\perp}$ vanish, as does $J_{\parallel\rightarrow\parallel}$ if $\hat{i}\perp\hat{s}$, where $\hat{s}$ is the unit vector in the observation direction. To this order, the other quantities are

$$J_{\perp\rightarrow\perp} = J_{\parallel\rightarrow\parallel}\cdot\frac{1}{(\hat{i}\cdot\hat{s})^2} = \frac{1}{(k_or)^2}(k_o^3 b^2 a)^2(m^2-1)^2\left(\frac{j_1(K)}{K}\right)^2$$

where k_o is the external propagation constant

r = distance to observation point from scatterer

a, b are semi-axes of spheroid

$K = \left| k\hat{i} - k_o\hat{s} \right| (b^2 \sin^2\psi + a^2\cos^2\psi)^{1/2}$

ψ = angle between $k\hat{i} - k_o\hat{s}$ and major axis

j_i is a spherical Bessel function

k is the internal propagation constant.

In the first order approximation, where two terms of the above series are used for the internal field, the quantities which vanish in the zero-order are given by the form

$$J_{a\to b} = \frac{1}{(k_o r)^2} m^2 (m^2 - 1)^4 (k_o b^2 a)^4 (X^2_{a\to b} + Y^2_{a\to b})$$

with $$X_{a\to b} = \mu_{m,n} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2m+3)(2n+3) \frac{j_{m+1}(k_o s')}{k_o s'} \frac{j_{n+1}(ki')}{ki'}$$

$$\cdot \frac{1}{4\pi} \int_{-1}^{1} dt_z \frac{1}{(kt')^2} j_{\mu>}(kt') n_{\mu<}(kt') \cdot \int_0^{2\pi} d\phi_T T^1_m(\cos\theta_s) T^1_n(\cos\theta_i)$$

$$\cdot (\hat{i}_a \cdot \hat{t}) \cdot (\hat{t} \cdot \hat{s}_b)$$

where $\mu_{m,n}$ is the parity modulus defined on p. 139.

$s' = \left| b\hat{s} + (a-b)\, s_z \hat{z} \right|$

$i' = \left| b\hat{i} + (a-b)\, i_z \hat{z} \right|$

$\hat{z}$ = unit vector in direction of axis of spheroid

i_z, s_z, t_z = components of vectors $\hat{i}$, $\hat{s}$, $\hat{t}$ to parallel to $\hat{z}$

$\hat{t}$ = unit vector $\parallel \underline{T}$

$\underline{T}$ = vector of integration point (dummy variable)

$\phi_T = \cos^{-1} t_z$

$t' = \left| b\hat{t} + (a-b)\, t_z \hat{z} \right|$

$\mu_> = \begin{matrix} m+1 \text{ for } m \geq n \\ n+1 \text{ for } n > m \end{matrix}$ and conversely for $\mu_<$

$T^1_m(\cos\theta_s)$ is a Gegenbauer function

$\theta_{i,s}$ = angle between $\hat{t}$ and $\hat{i}$, $\hat{s}$

$$\hat{i}_{\perp} = \hat{s}_{\perp} = \frac{(\hat{s} \wedge \hat{i})}{\sqrt{1-(\hat{i} \cdot \hat{s})^2}} = \text{unit vector} \perp \text{observation plane}$$

$$\hat{s}_{||} = -\frac{(\hat{s} \wedge (\hat{s} \wedge \hat{i}))}{\sqrt{1-(\hat{i} \cdot \hat{s})^2}} = \text{unit vector in observation plane} \perp \hat{s}$$

$$\hat{i}_{||} = \frac{(\hat{i} \wedge (\hat{i} \wedge \hat{s}))}{\sqrt{1-(\hat{i} \cdot \hat{s})^2}} = \text{unit vector in observation plane} \perp \hat{i}$$

and finally

$$Y_{a \rightarrow b} = \frac{1}{4\pi} \int_{-1}^{1} dt_z \cdot \int_0^{2\pi} d\phi_T \, \frac{j_1(K_s^o)}{K_s^o} \cdot \frac{j_1(K_i^o)}{K_i^o} (\hat{i}_a \cdot \hat{t})(\hat{t} \cdot \hat{s}_b)$$

where

$$K_i^o = k\left| \underline{i}' - \underline{t}' \right|$$

$$K_s^o = \left| k_o \underline{s}' - k \underline{t}' \right|$$

$$\underline{i}' = b\hat{i} + (a-b)\, i_z \hat{z}$$

$$\underline{s}' = b\hat{s} + (a-b)\, s_z \hat{z}$$

$$\underline{t}' = b\hat{t} + (a-b)\, t_z \hat{z}$$

Comments on range of validity appear in the text.

4.2 THEORETICAL CURVES

The following are the graphical representations of the principal numerical results obtained to date from the analyses described in the preceding chapters.

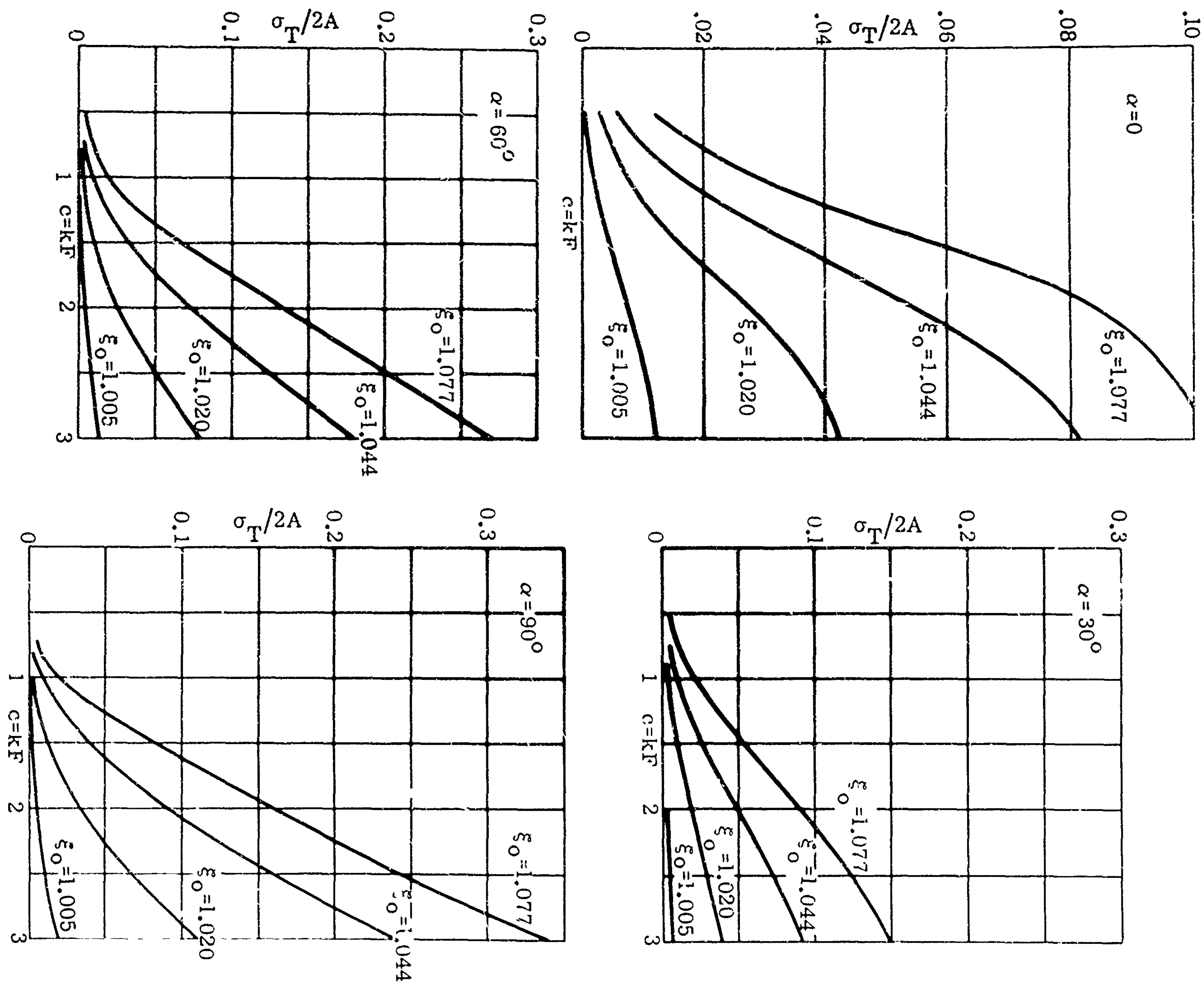


FIG. 4: RELATIVE CROSS SECTIONS OF HARD SPHEROIDS (Spence and Granger, 1951)
σ_T = total scattering cross section; A = area of geometrical shadow, α = angle between incident direction and major axis.

$$A = \pi F^2 \left[\frac{\xi_o^2 - 1}{\xi_o^2 - \cos^2\alpha}\right]^{1/2} \cdot \left[\xi_o^2 \sin^2\alpha + (\xi_o^2 - 1)\cos^2\alpha\right]$$

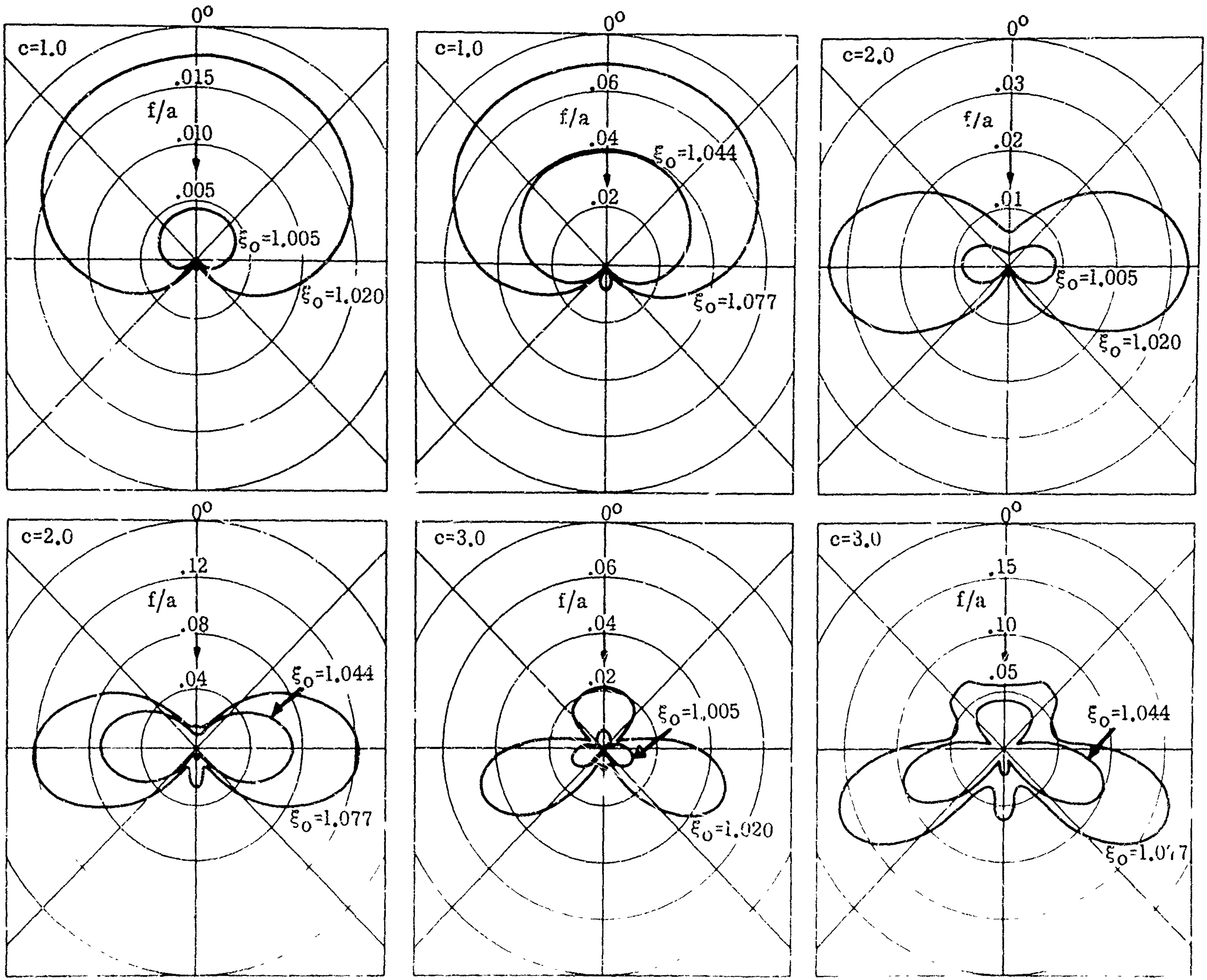


FIG. 5: SCATTERING PATTERNS FOR HARD SPHEROIDS, PLANE WAVE INCIDENT NOSE-ON (Spence and Granger, 1951) $f \equiv \lim\ r e^{-ikr} \cdot \psi^s$, $\psi^s \equiv$ scattered field.

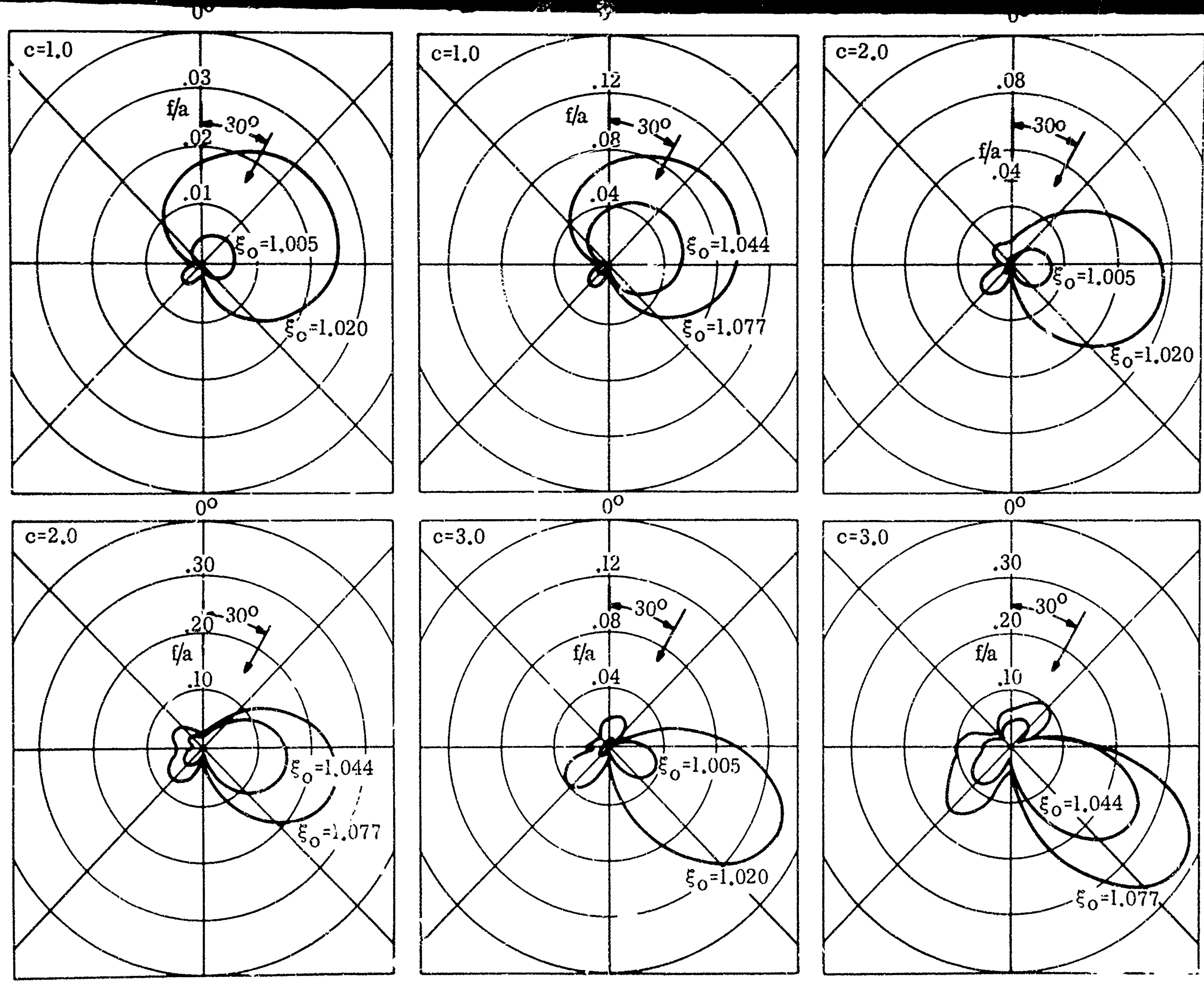


FIG. 6: SCATTERING PATTERNS IN PLANE OF INCIDENCE FOR HARD SPHEROIDS, PLANE WAVE INCIDENT 30° OFF NOSE (Spence and Granger, 1951). $f \equiv \lim_{r \to \infty} r e^{-ikr} \psi^s$, ψ^s scattered field.

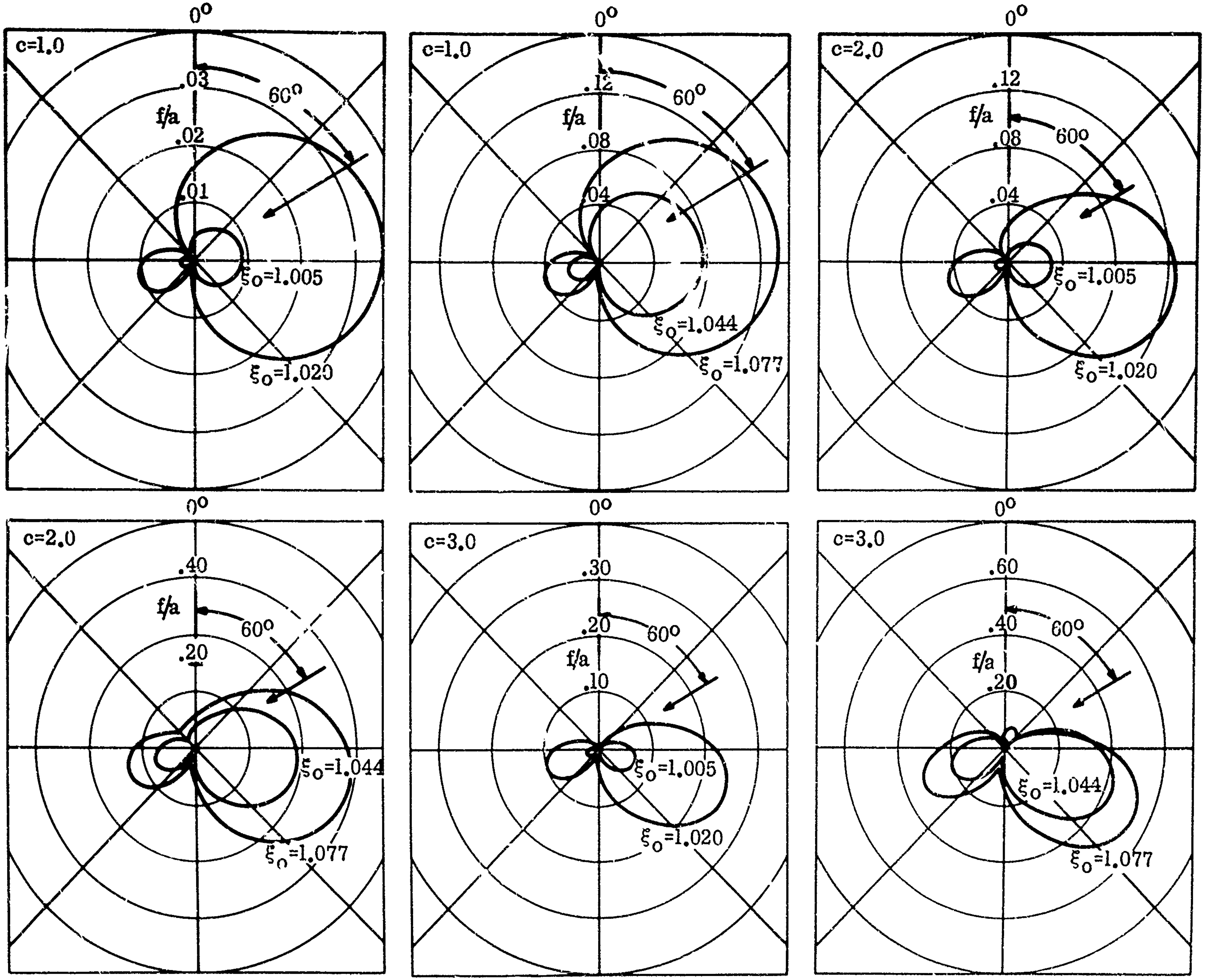


FIG. 7: SCATTERING PATTERNS IN PLANE OF INCIDENCE FOR HARD SPHEROIDS, PLANE WAVE INCIDENT 60° OFF NOSE (Spence and Granger, 1951). $f \equiv \lim_{r \to \infty} r\, e^{-ikr}\, \psi^s$, $\psi^s \equiv$ scattered field.

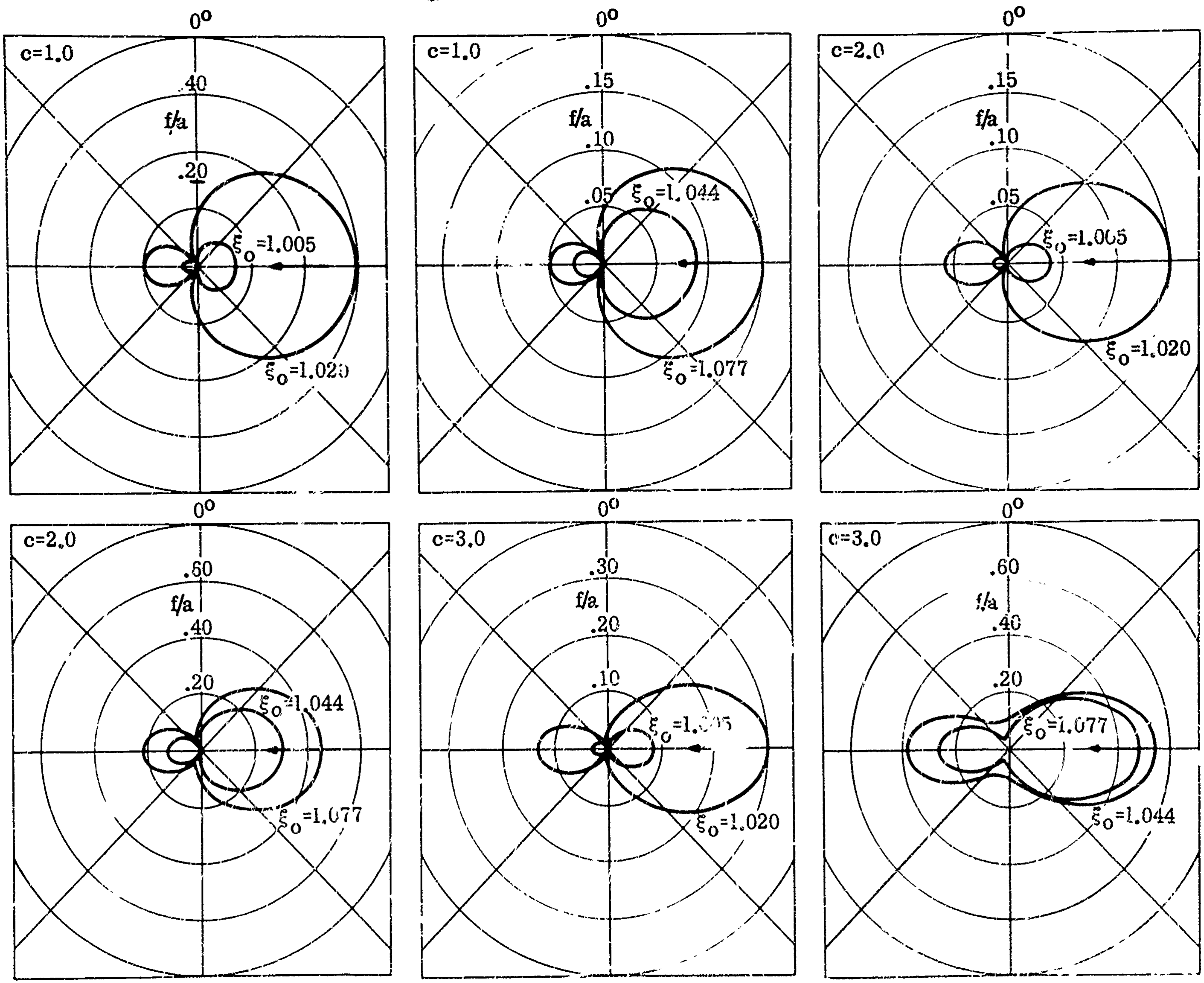


FIG. 8: SCATTERING PATTERNS IN PLANE OF INCIDENCE FOR HARD SPHEROIDS, PLANE WAVE INCIDENT 90° OFF NOSE (Spence and Granger, 1951). $f = \lim_{r \to \infty} r e^{-ikr} \psi^s$, ψ^s = scattered field.

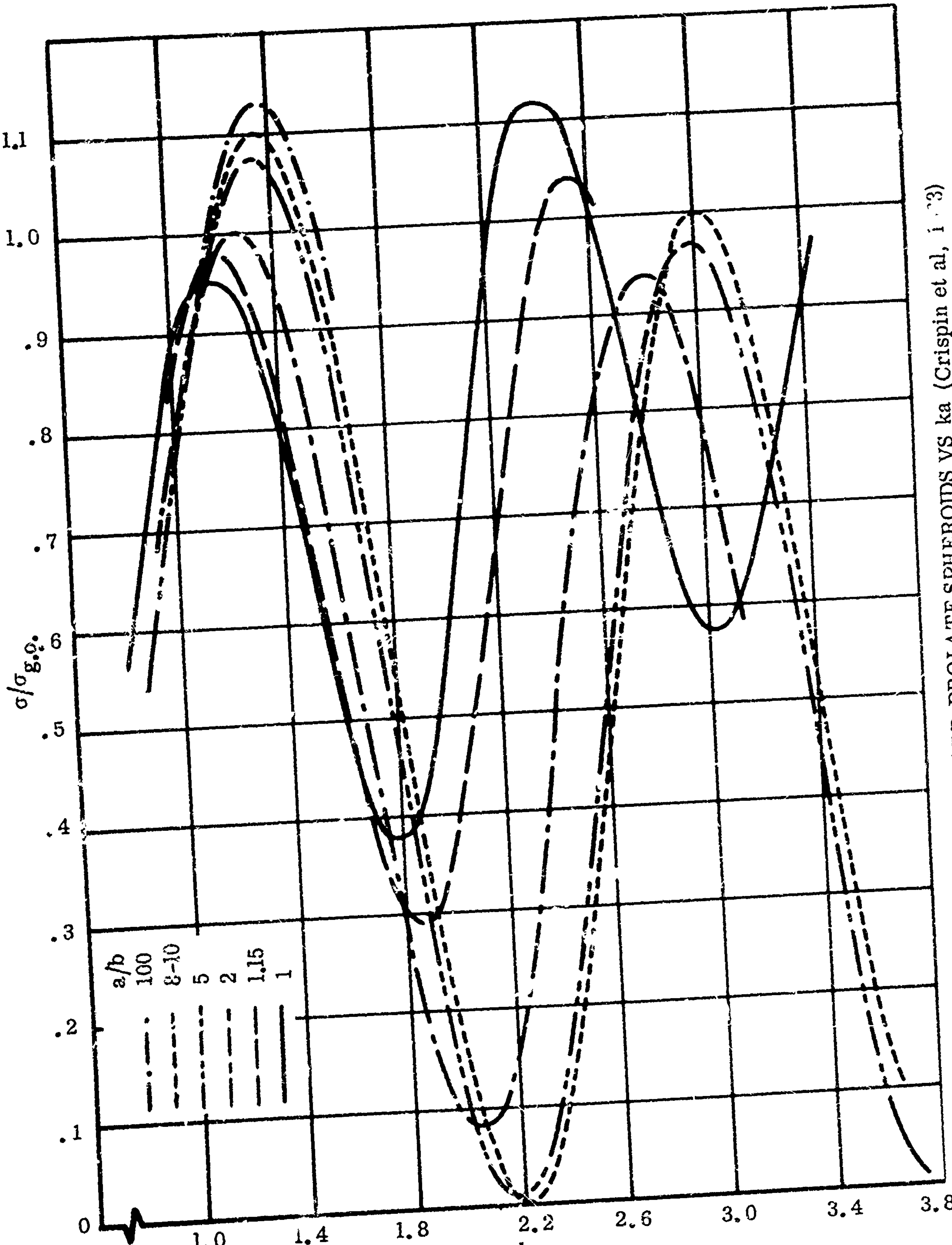


FIG. 9: SCALAR CROSS SECTIONS OF HARD PROLATE SPHEROIDS VS ka (Crispin et al, 1 3)

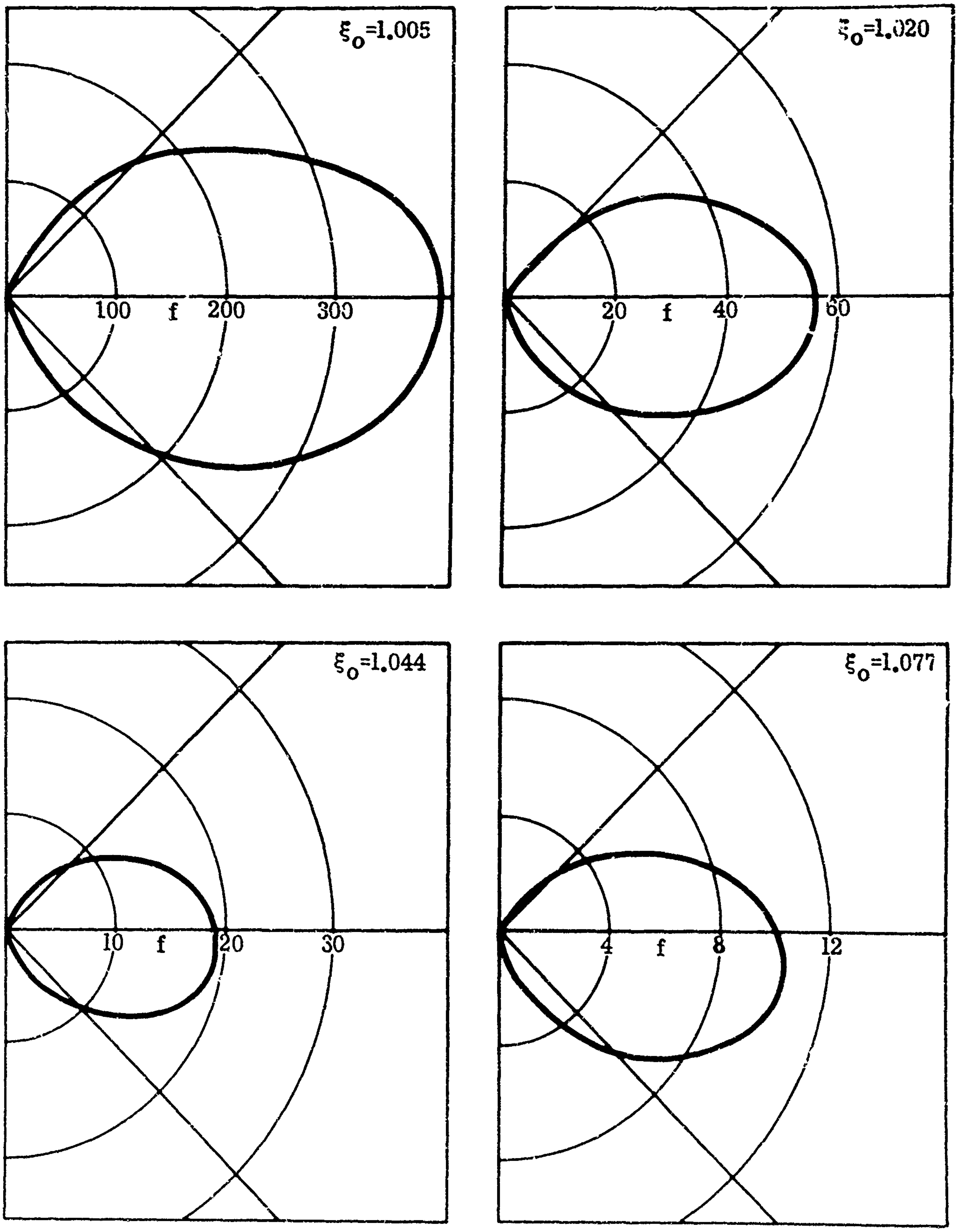


FIG. 10: RELATIVE POWER RADIATED FROM AN AXIAL DIPOLE AT THE TIP OF A PROLATE SPHEROID, ka=1.0 (Hatcher and Leitner, 1954).

$$f = \pi^2 a^4 \epsilon \mu \left| \lim_{r \to \infty} r H_\phi \right|^2$$

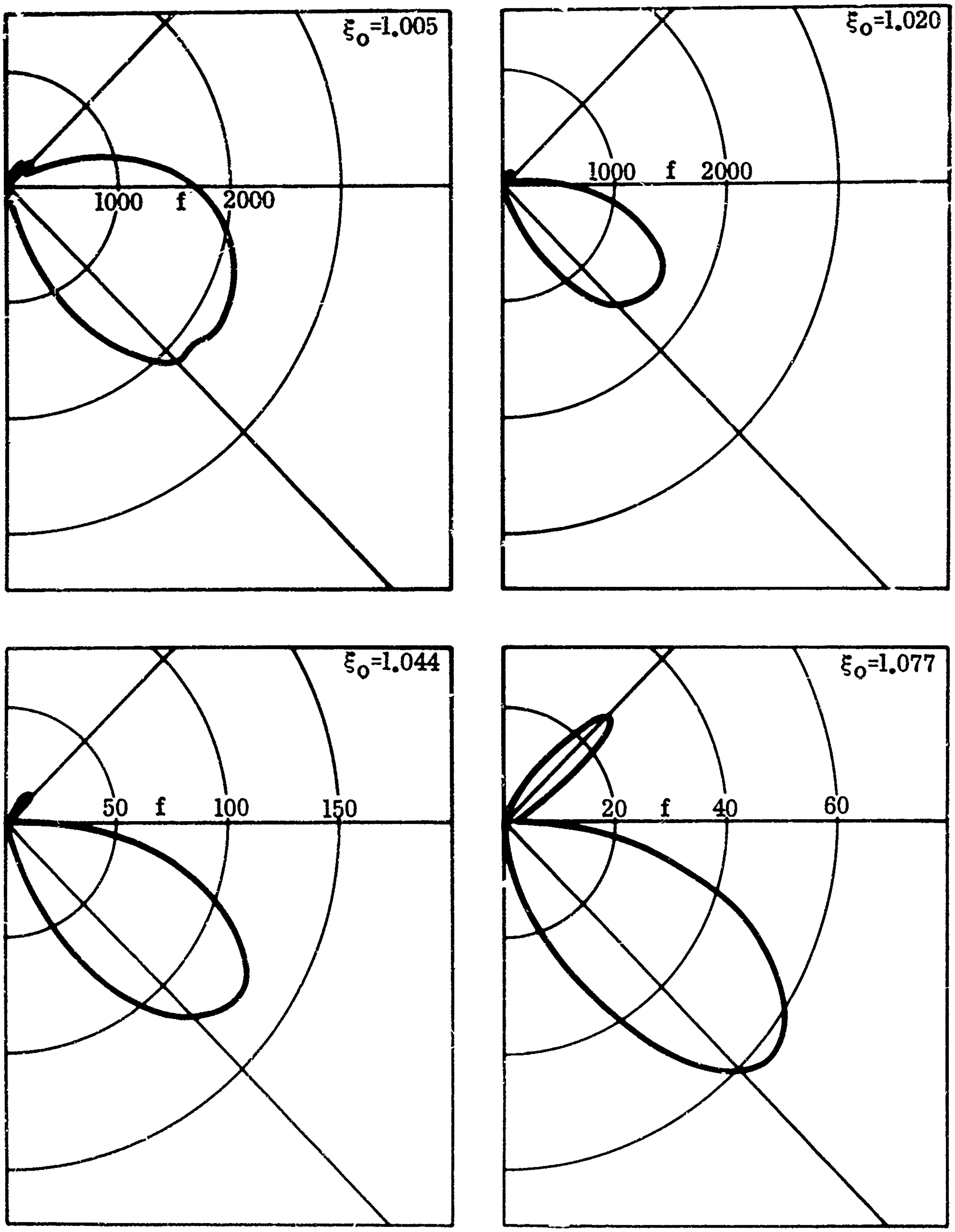


FIG. 11: RELATIVE POWER RADIATED FROM AN AXIAL DIPOLE AT THE TIP OF A PROLATE SPHEROID, ka=2.0 (Hatcher and Leitner, 1954)
$f = \pi^2 a^4 \epsilon \mu \left| \lim_{r \to \infty} r H_\phi \right|^2$

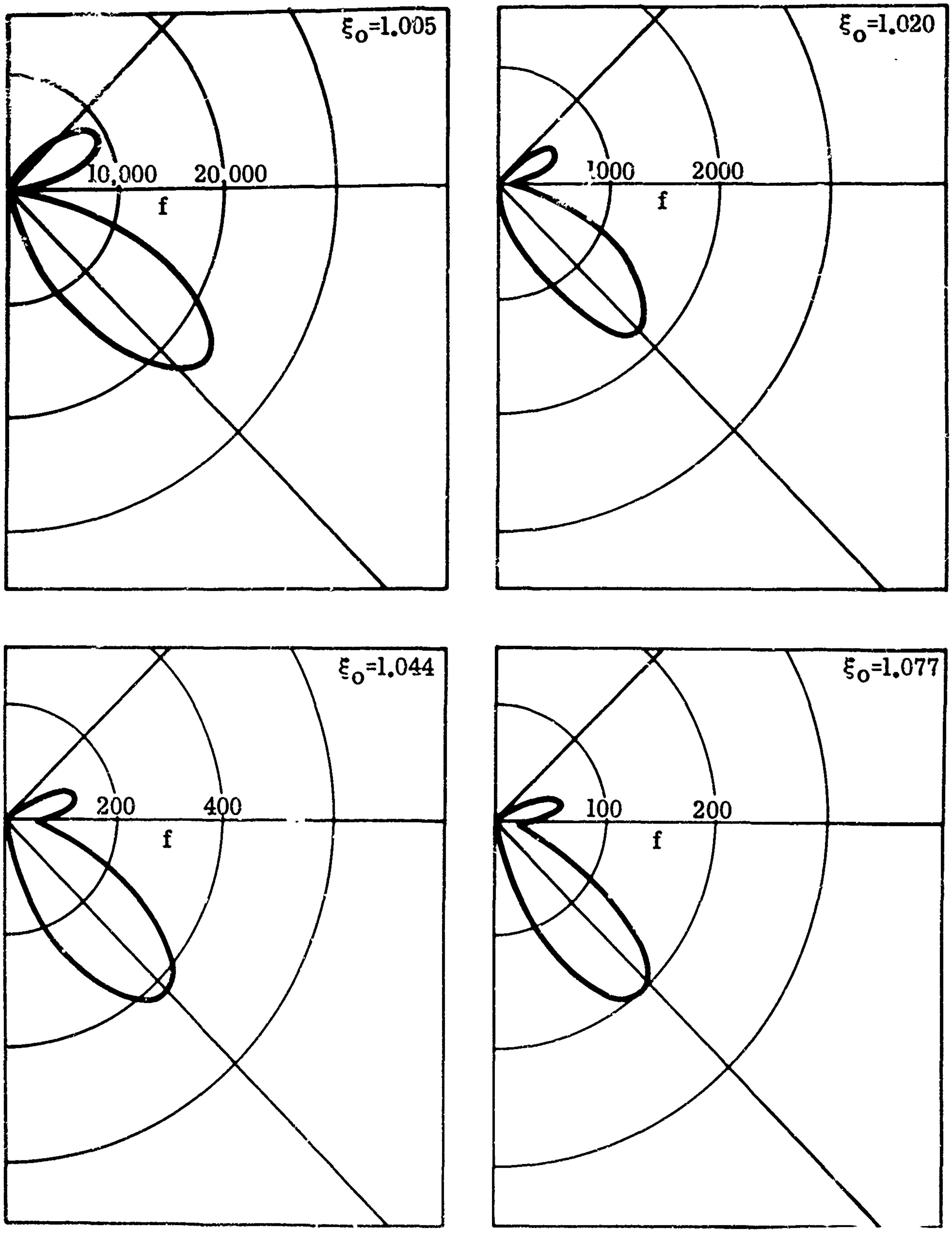


FIG. 12: RELATIVE POWER RADIATED FROM AN AXIAL DIPOLE AT THE TIP OF A PROLATE SPHEROID, ka=3.0 (Hatcher and Leitner, 1954).

$$f = \pi^2 a^4 \epsilon \mu \left| \lim_{r \to \infty} r H_\phi \right|^2$$

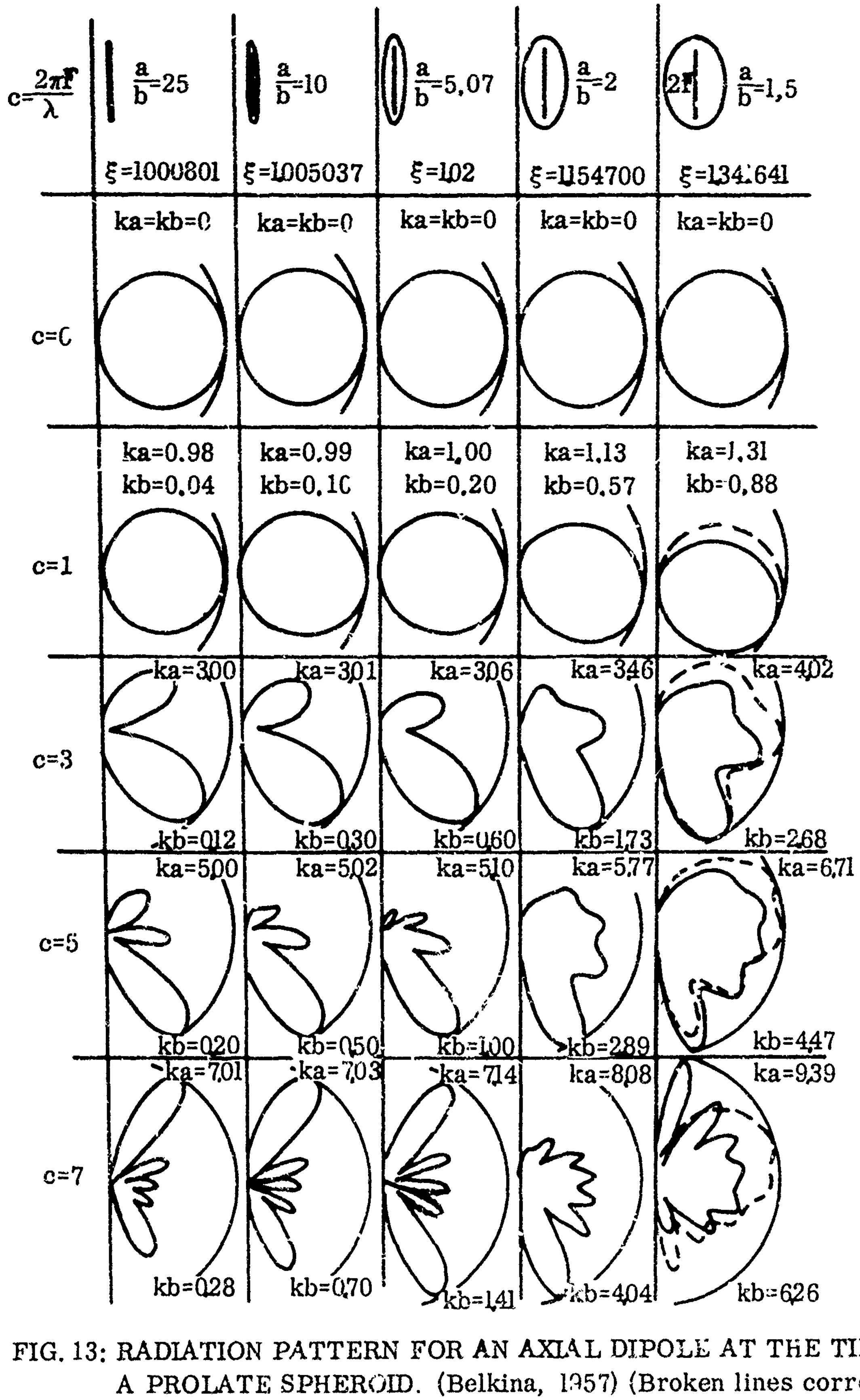


FIG. 13: RADIATION PATTERN FOR AN AXIAL DIPOLE AT THE TIP OF A PROLATE SPHEROID. (Belkina, 1957) (Broken lines correspond to sphere of radius r = c/k.)

(a) Exact Solution (Siegel et al, 1956)
(b) Power Series, 1 Term (Rayleigh, 1898)
(c) Power Series, 2 Terms (Stevenson, 1953)
(d) Power Series, 3 Terms (corrected, see footnote, p.60)

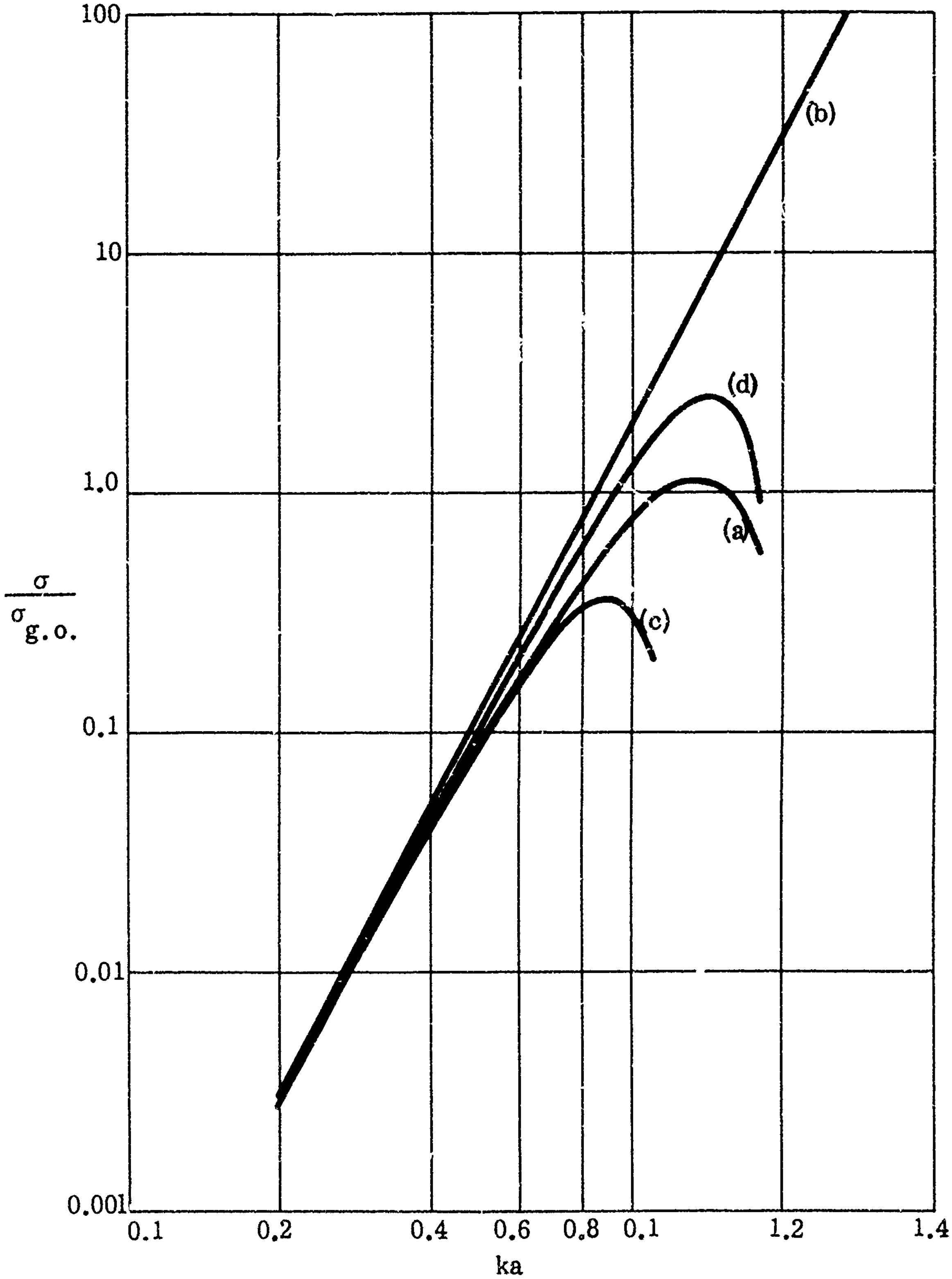


FIG. 14: LOW FREQUENCY NOSE-ON BACK SCATTERED CROSS SECTION FROM A HARD 10:1 PROLATE SPHEROID (Sleator, 1960)

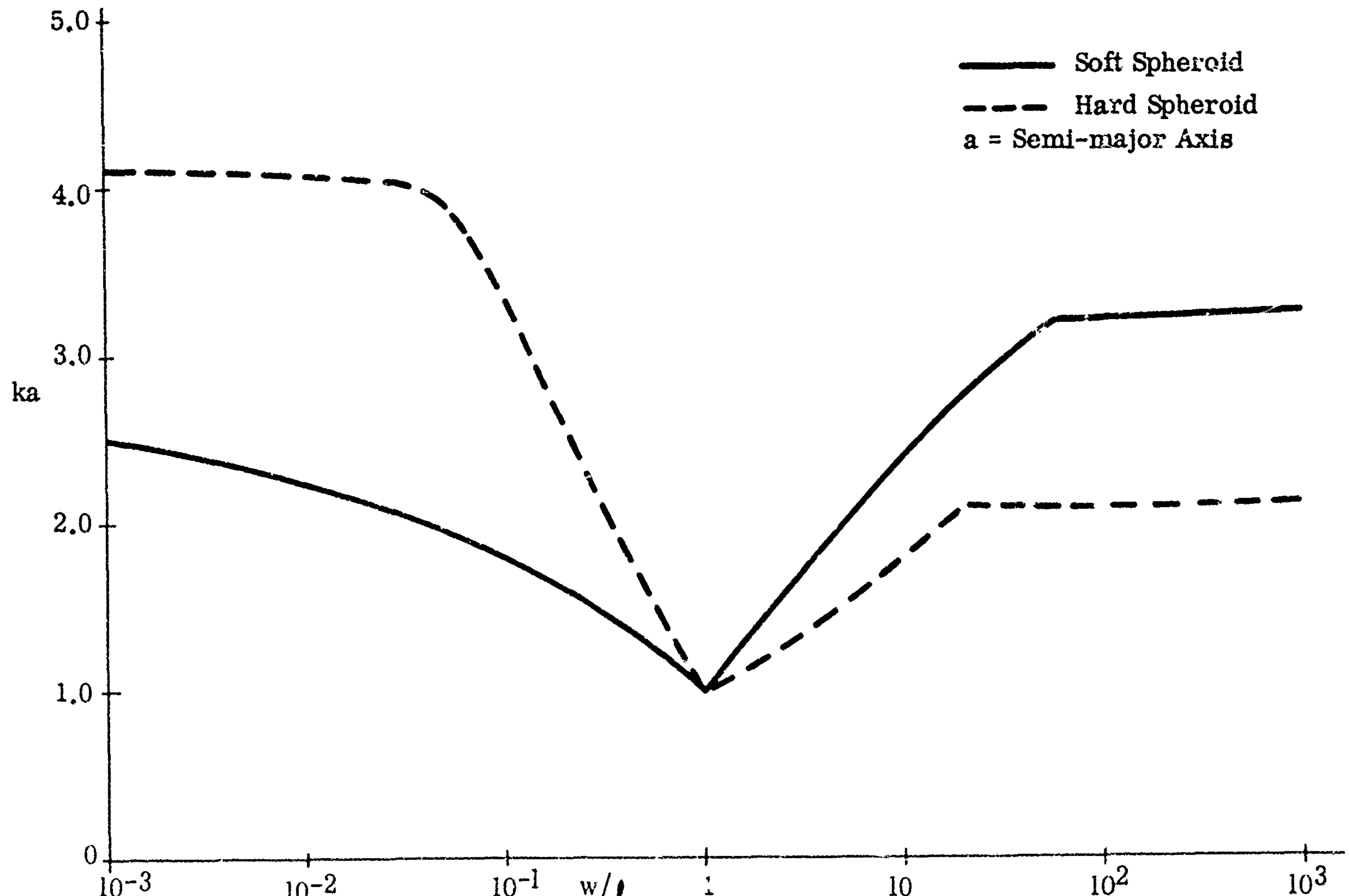


FIG. 15: RADIUS OF CONVERGENCE OF RAYLEIGH SERIES (Senior, 1961). w and ℓ are the dimensions perpendicular and parallel to the direction of the incident field.

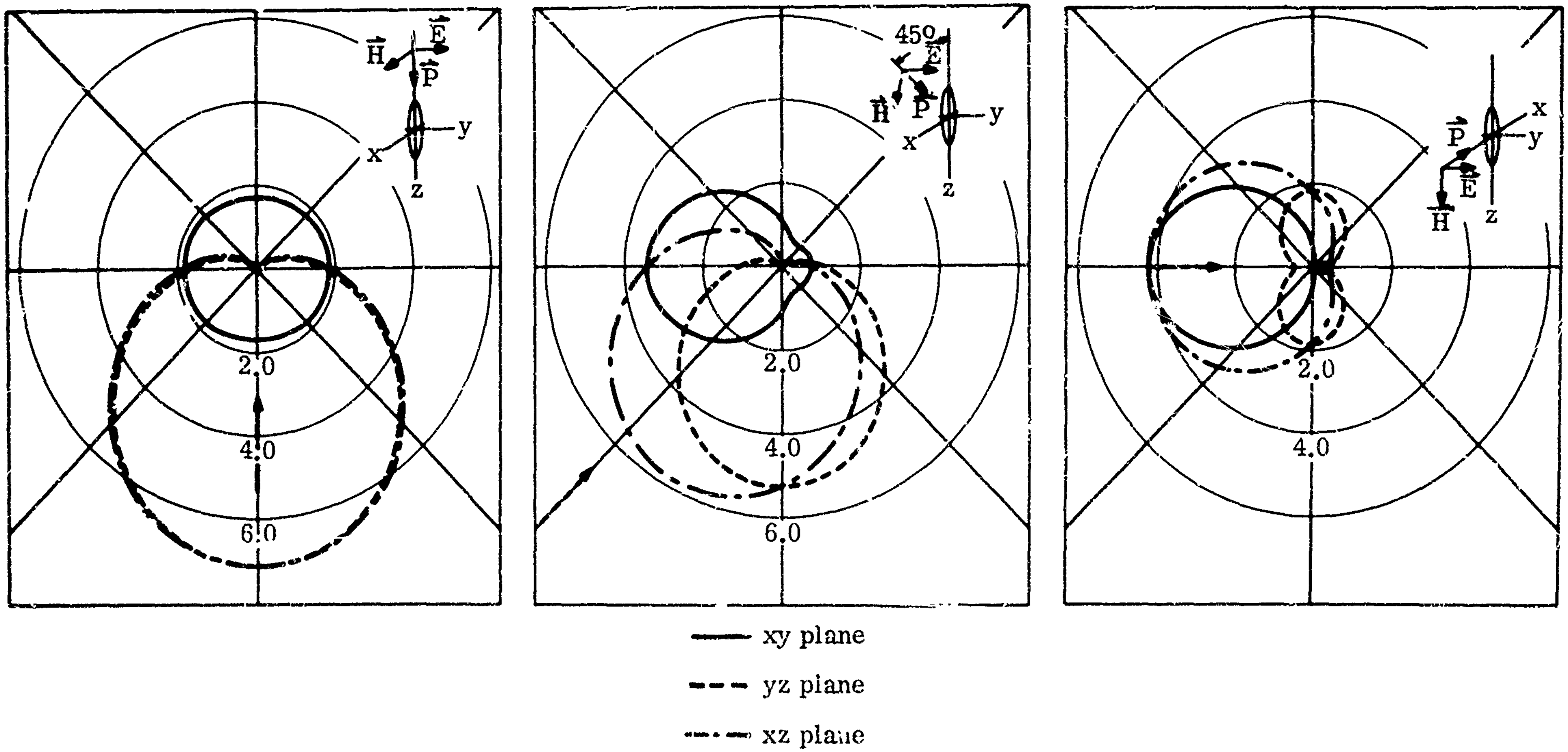


FIG. 16: RAYLEIGH SCATTERING OF A PLANE WAVE BY A 10:1 SPHEROID: COEFFICIENT OF $(ka)^4$ IN SCATTERING CROSS SECTION FOR THREE ORTHOGONAL PLANES (Sleator, 1959).

(Note: Fig. 16 is continued on next page.)

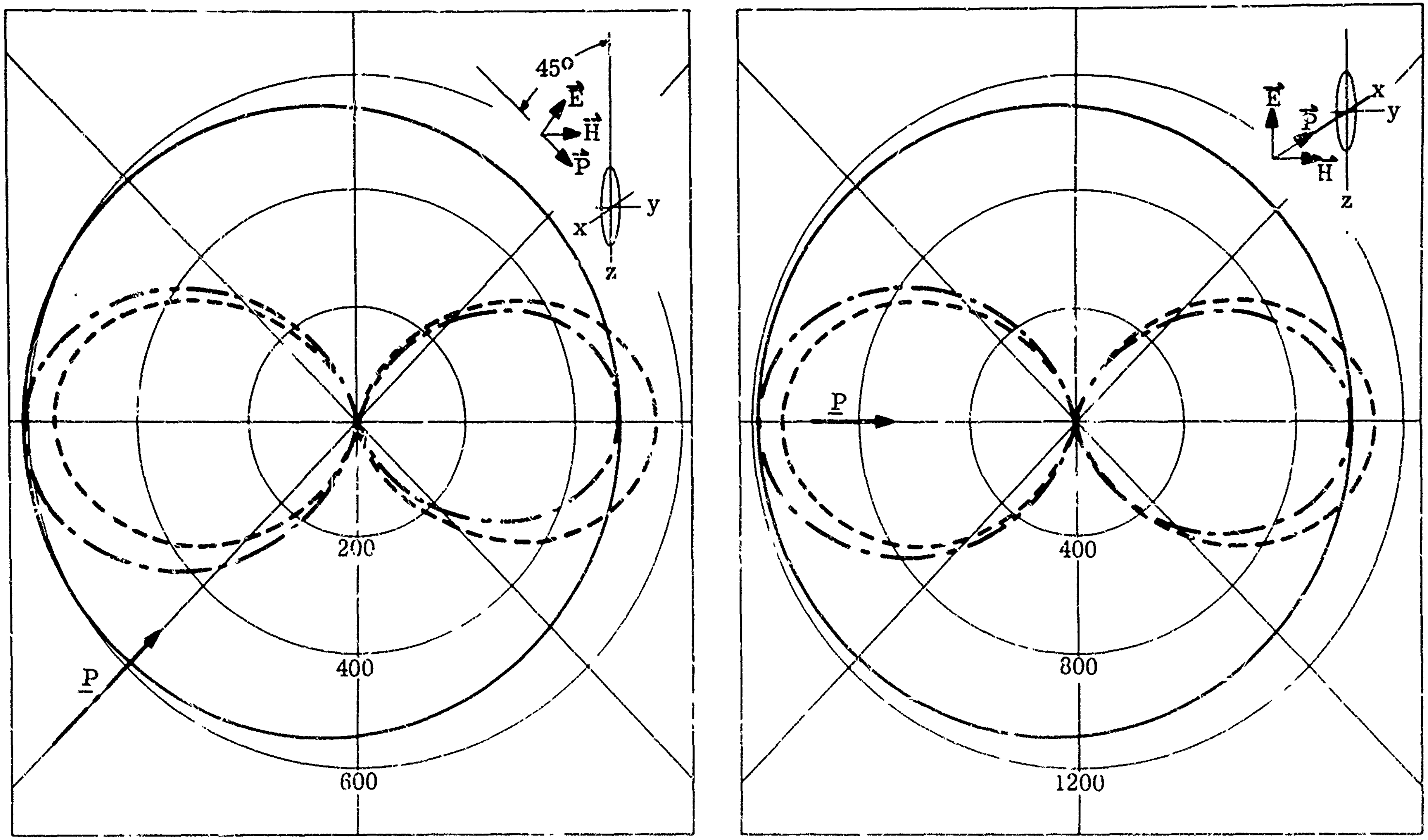
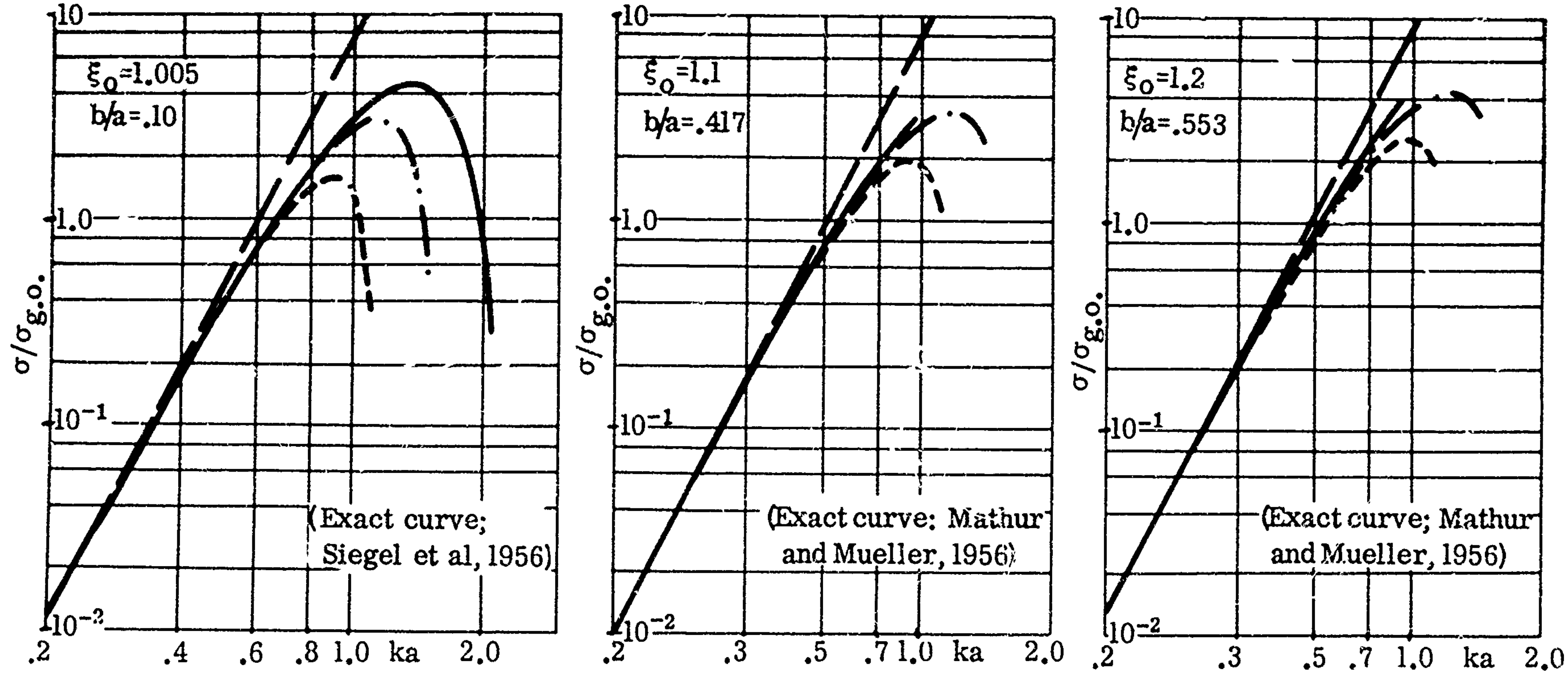


FIG. 16: RAYLEIGH SCATTERING (continued).



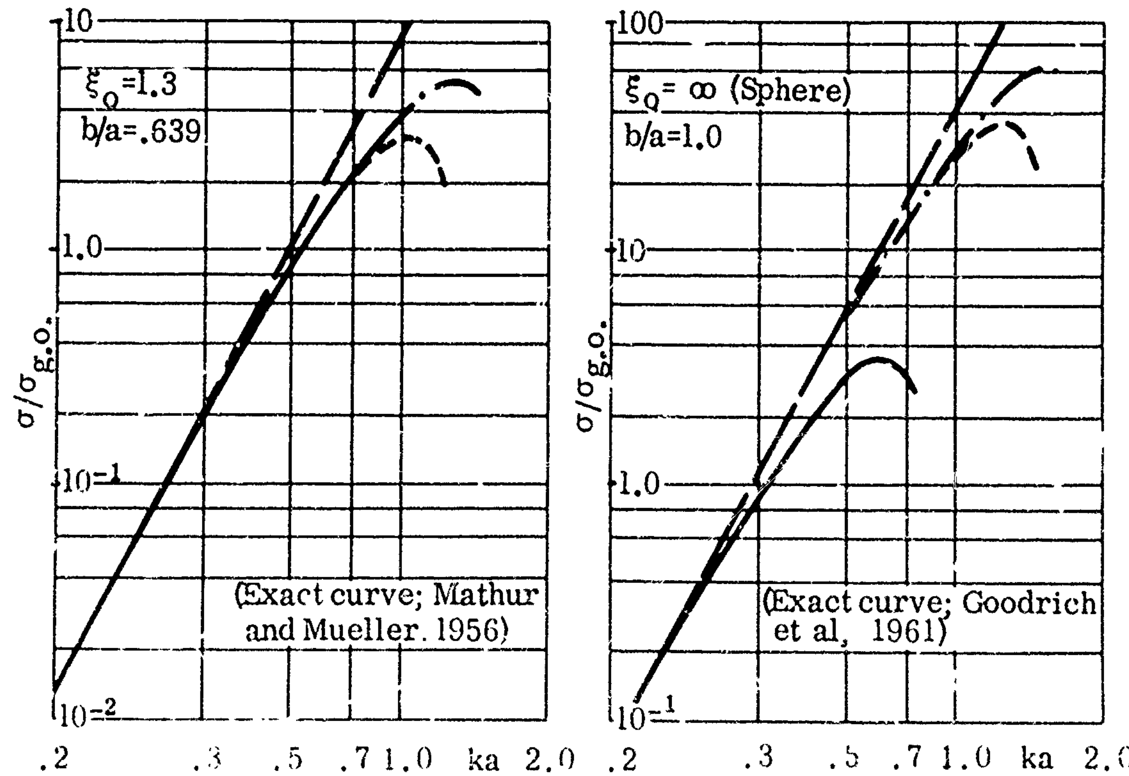


FIG. 17: ELECTROMAGNETIC BACK SCATTERING FROM A PROLATE SPHEROID WITH PLANE WAVE INCIDENT NOSE-ON

— — — Rayleigh Law
———— Exact Curve
Stevenson Solution:
—·—·— $\sigma/\sigma_{g.o.} = 4\left[A(ka)^2 + B(ka)^4\right]^2$
– – – – – $\sigma/\sigma_{g.o.} = 4A^2(ka)^4 + 8AB(ka)^6$

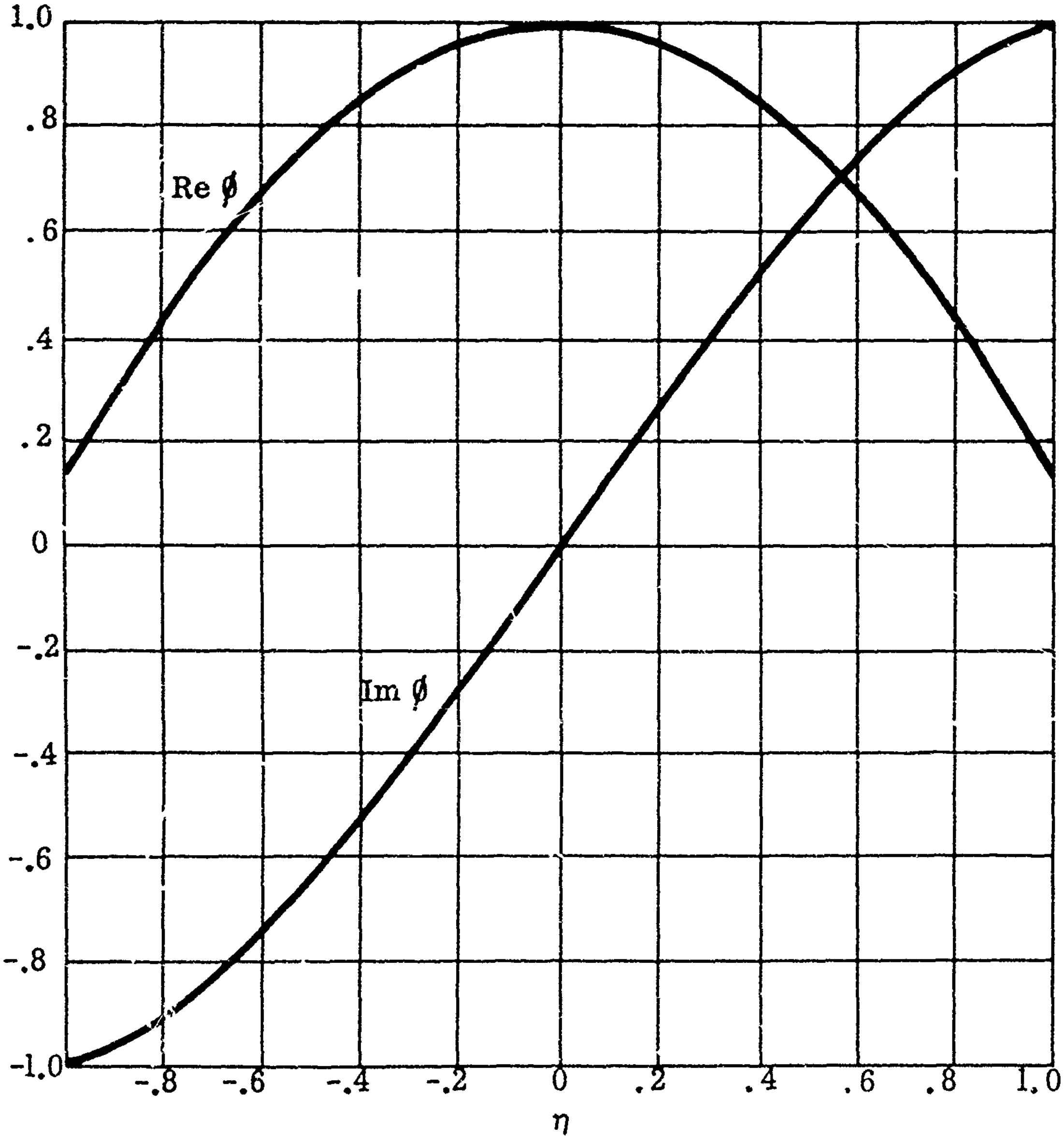


FIG. 18: POTENTIAL DISTRIBUTION ON THE SURFACE OF A HARD 10:1 PROLATE SPHEROID WITH PLANE WAVE INCIDENT NOSE-ON. $\phi = e^{-ikz} + \phi^s$ (Sleator, 1960).

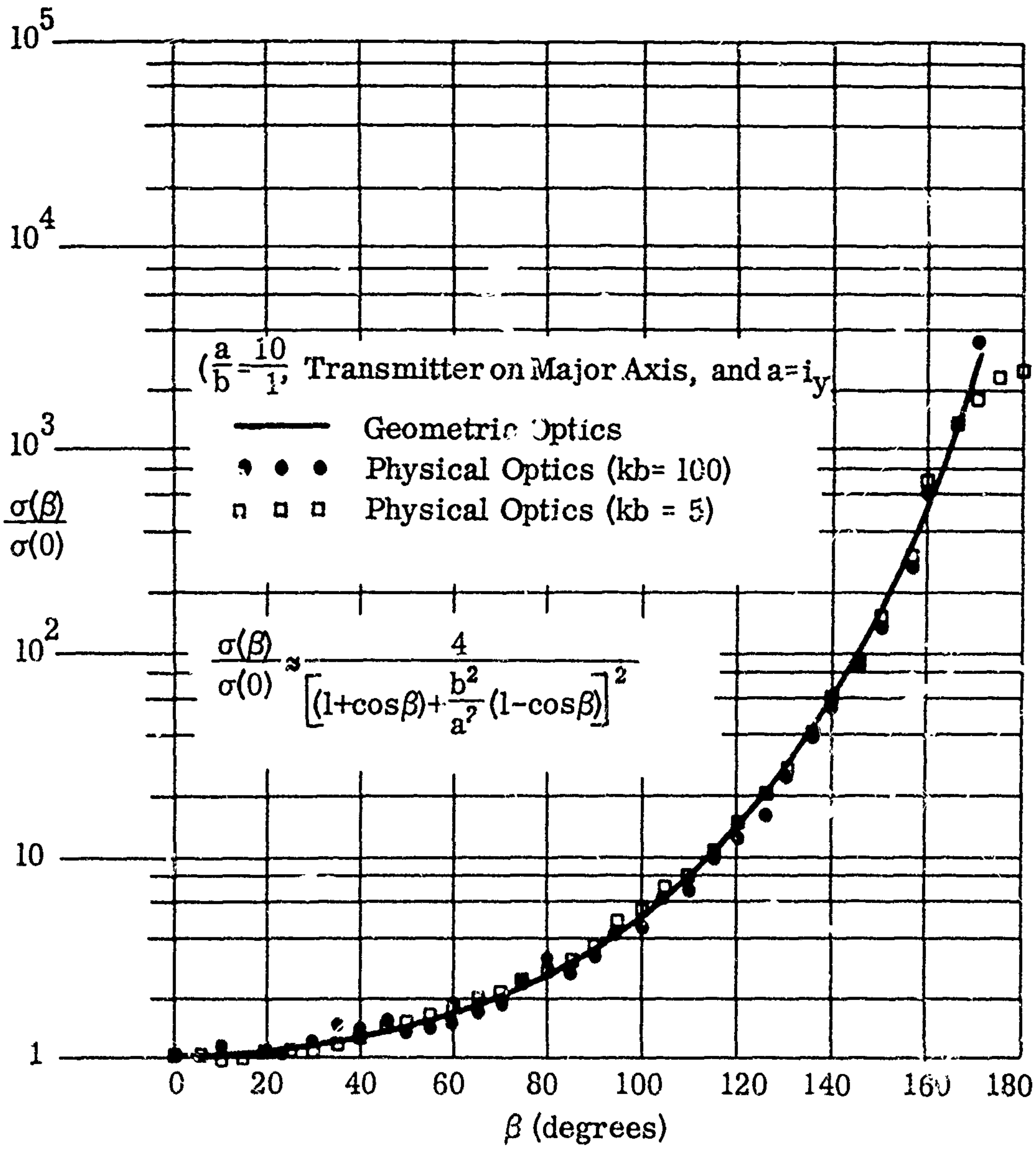


FIG. 19: PHYSICAL OPTICS CROSS SECTION OF A PROLATE SPHEROID AS A FUNCTION OF SEPARATION ANGLE β. (Siegel et al, 1955a)

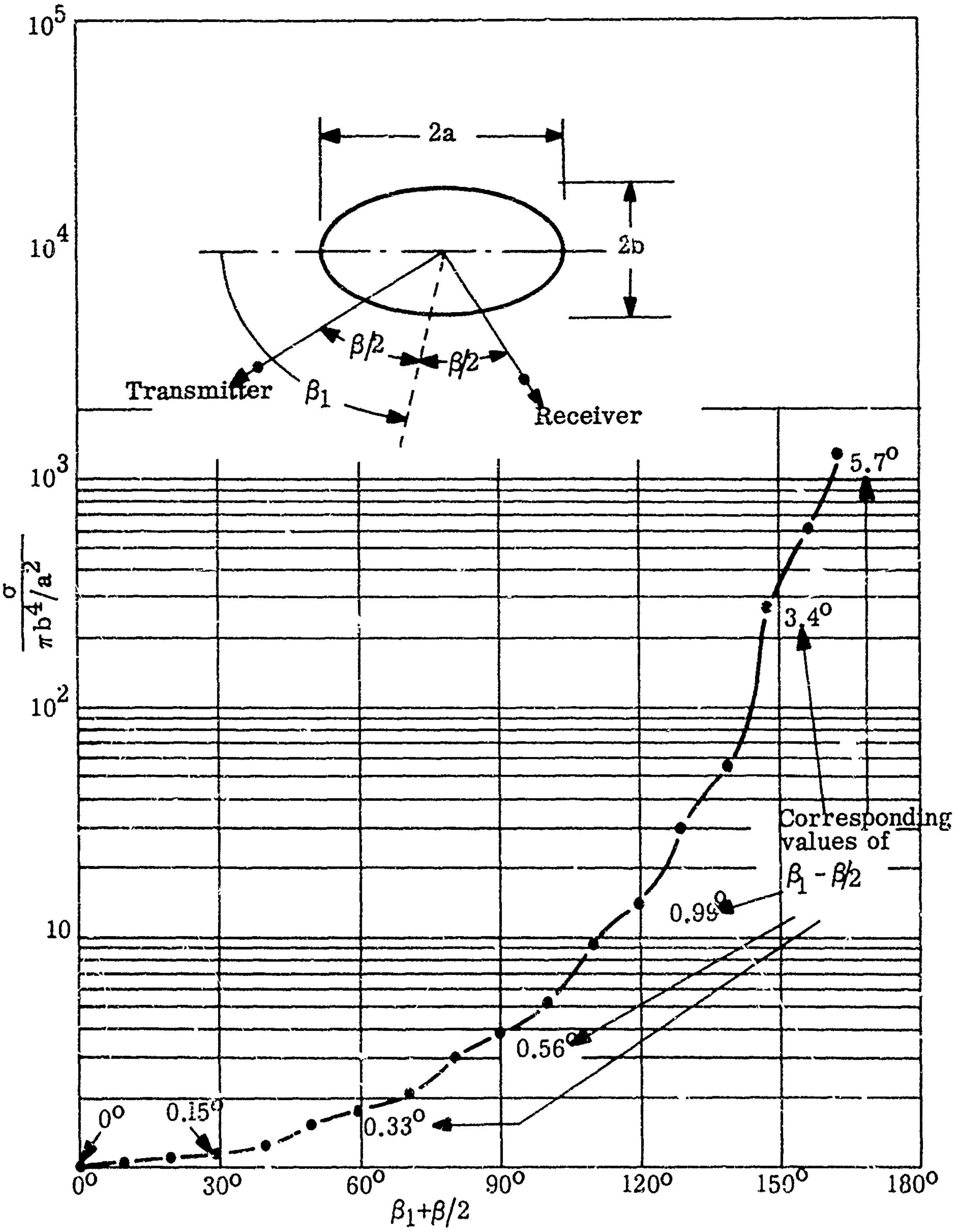
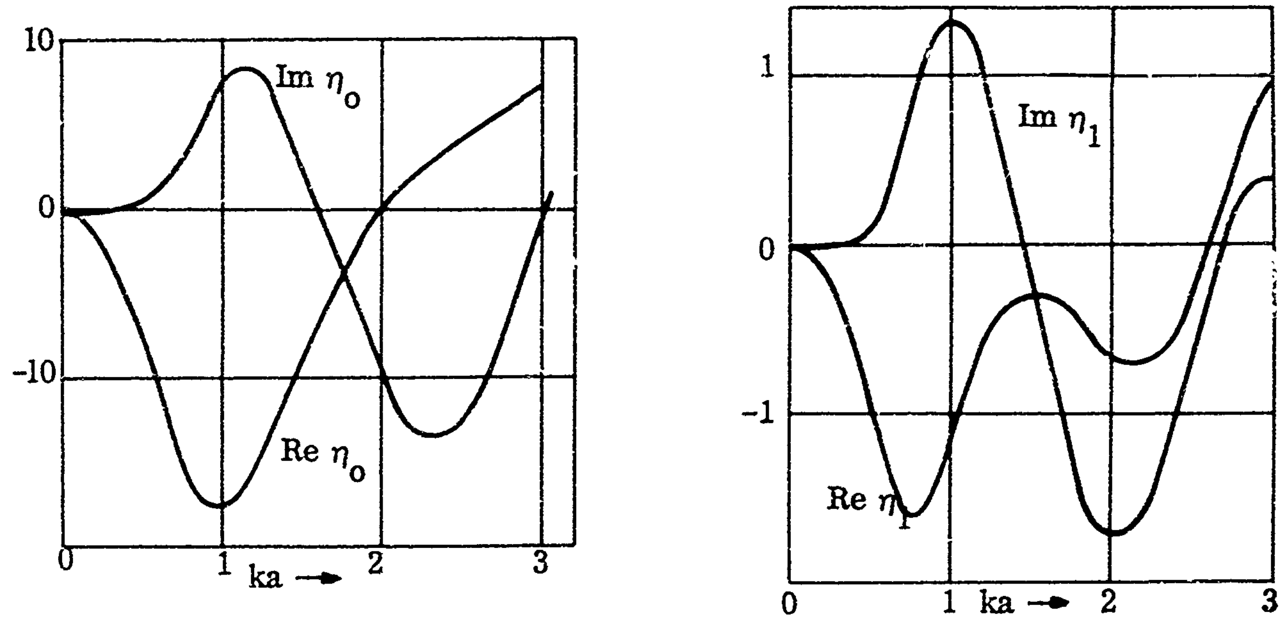


FIG. 20: BISTATIC CROSS SECTION OF A PROLATE SPHEROID. EXACT PHYSICAL OPTICS RESULT FOR a/b=10, ka=25. (Siegel et al, 1955a)



$$\frac{\sigma}{\pi a^2} = \left| \eta_o + (1 - a^2/b^2)\eta_1 \right|^2$$

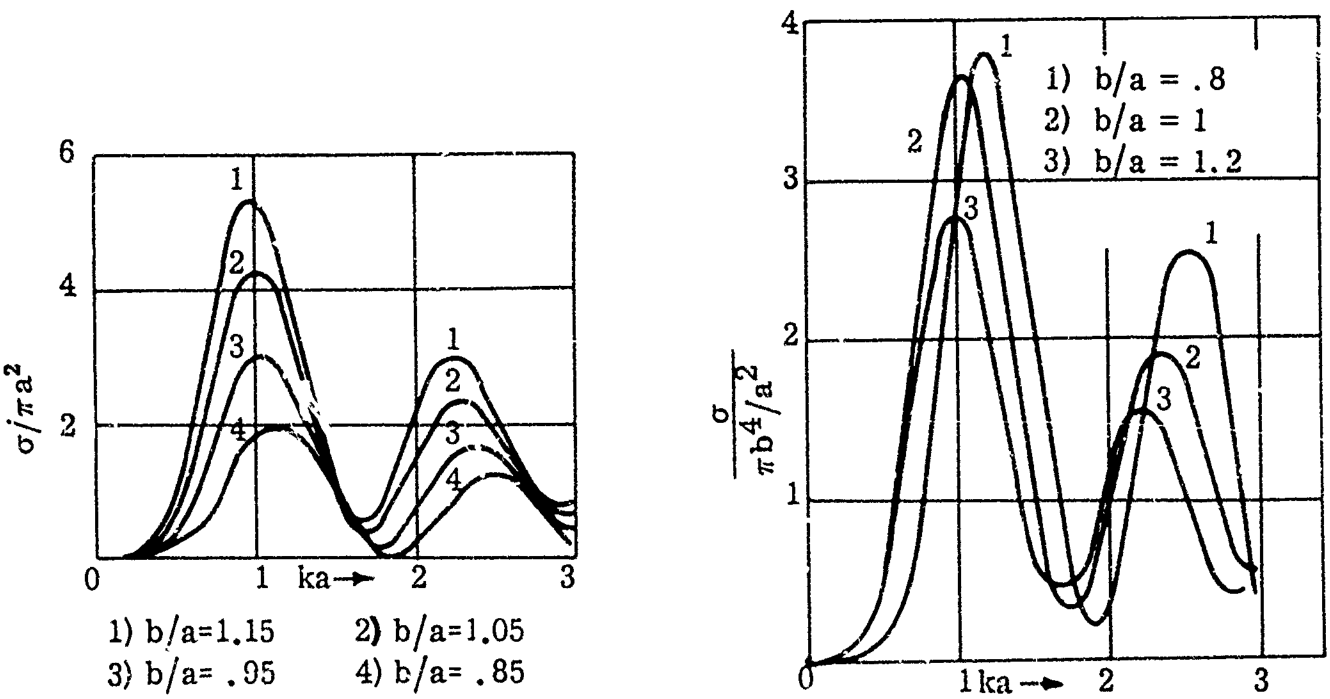
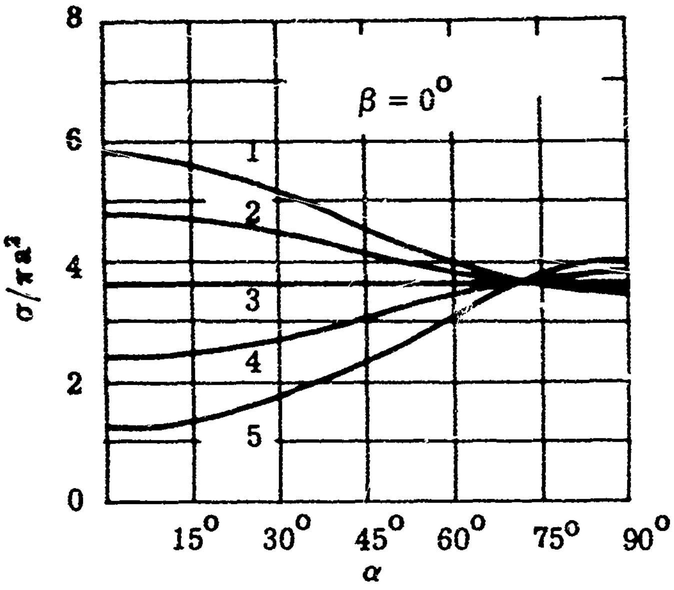


FIG. 21: NOSE-ON BACK SCATTERING CROSS SECTION OF SPHEROIDS WITH SMALL ECCENTRICITY
(Mushiake, 1956).

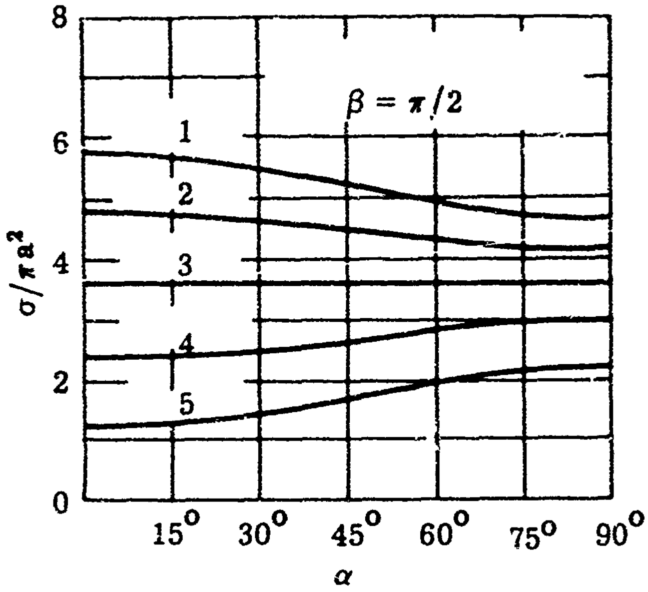
α = angle between axis of symmetry and direction of incidence

β = angle between plane of α and incident $\underline{E}^i$

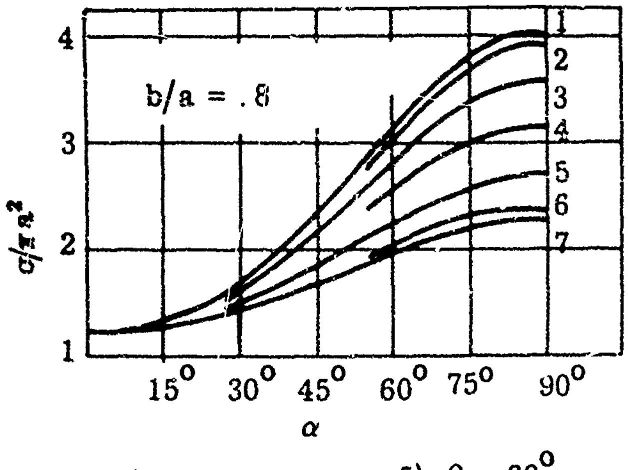
$$\underline{E}^i = E^i\,(\hat{i}_x \cos\alpha \cos\beta + \hat{i}_y \sin\beta - \hat{i}_z \sin\alpha \cos\beta)$$



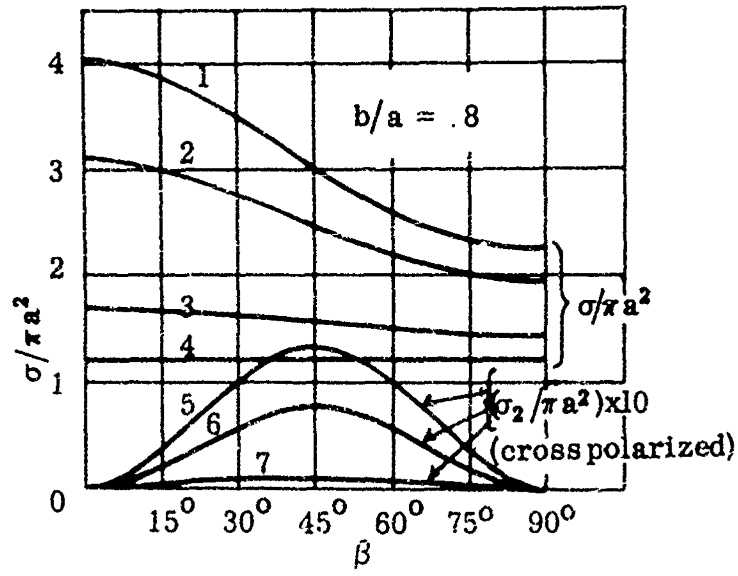
1) b/a = 1.2
2) b/a = 1.1
3) b/a = 1.0
4) b/a = .9
5) b/a = .8



1) b/a = 1.2
2) b/a = 1.1
3) b/a = 1.0
4) b/a = .9
5) b/a = .8



1) $\beta = 0$
2) $\beta = 15^0$
3) $\beta = 30^0$
4) $\beta = 45^0$
5) $\beta = 60^0$
6) $\beta = 75^0$
7) $\beta = 90^0$



1) $\alpha = 90^0$
2) $\alpha = 60^0$
3) $\alpha = 30^0$
4) $\alpha = 0^0$
5) $\alpha = 90^0$
6) $\alpha = 60^0$
7) $\alpha = 30^0$

FIG. 22: BACK SCATTERED CROSS SECTION OF SPHEROIDS AS A FUNCTION OF ANGLE OF INCIDENCE AND POLARIZATION FOR ka = 1 (Mushiake, 1956)

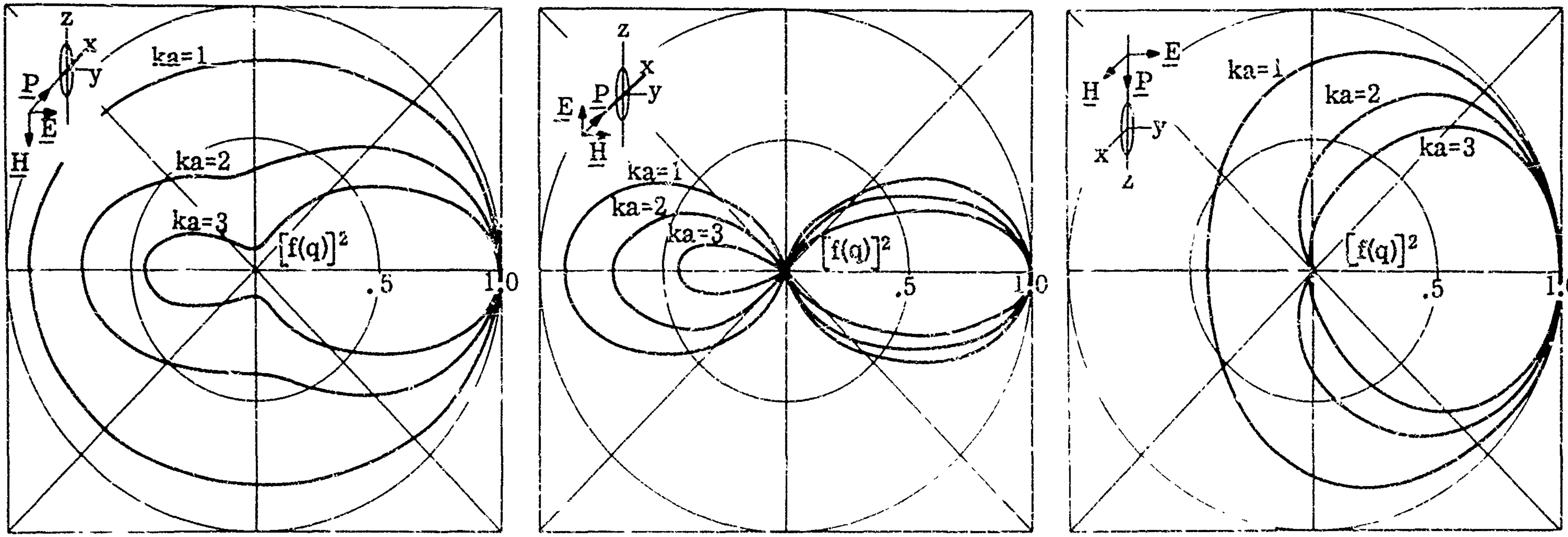


FIG. 23: RELATIVE BISTATIC CROSS SECTION OF A DIELECTRIC SPHEROID IN PLANE OF SYMMETRY AXIS AND INCIDENT DIRECTION.
$a/b = 3$, $f(q)$ as in Eq. (3.82) (Shatilov, 1960).

4.3 EXPERIMENTAL RESULTS

In view of the fact that the prolate spheroid has been the object of a considerable amount of theoretical investigation, it is rather surprising to find that spheroids are not nearly so popular as objects of experimental study. This, in part, is a result of the difficulty incumbent upon measurements involving low cross section shapes, in which category the prolate spheroid often falls. It would seem, however, in view of the considerable interest in the scattering properties of spheroids and the increased measurement capabilities of various laboratories, that a comprehensive program of experimental measurements would be well justified at this point if one has not already been begun.

At the present writing, the list of experimental studies on the prolate spheroid is short and the available data are quite limited. As an illustration of the scarcity of these data, Fig. 24 depicts all available back scattering data for that case where data are most plentiful, i.e., nose-on back scattering from a conducting prolate spheroid with major to minor axis ratio of 10:1. Also included in the figure are available theoretical results. This assessment of experimental work is based on a study of the published literature as well as private communications which are enumerated below. Any omissions are inadvertent and it would be greatly appreciated if such data were communicated to the Radiation Laboratory. All of the work discussed in this section concerns the electromagnetic (vector) case.

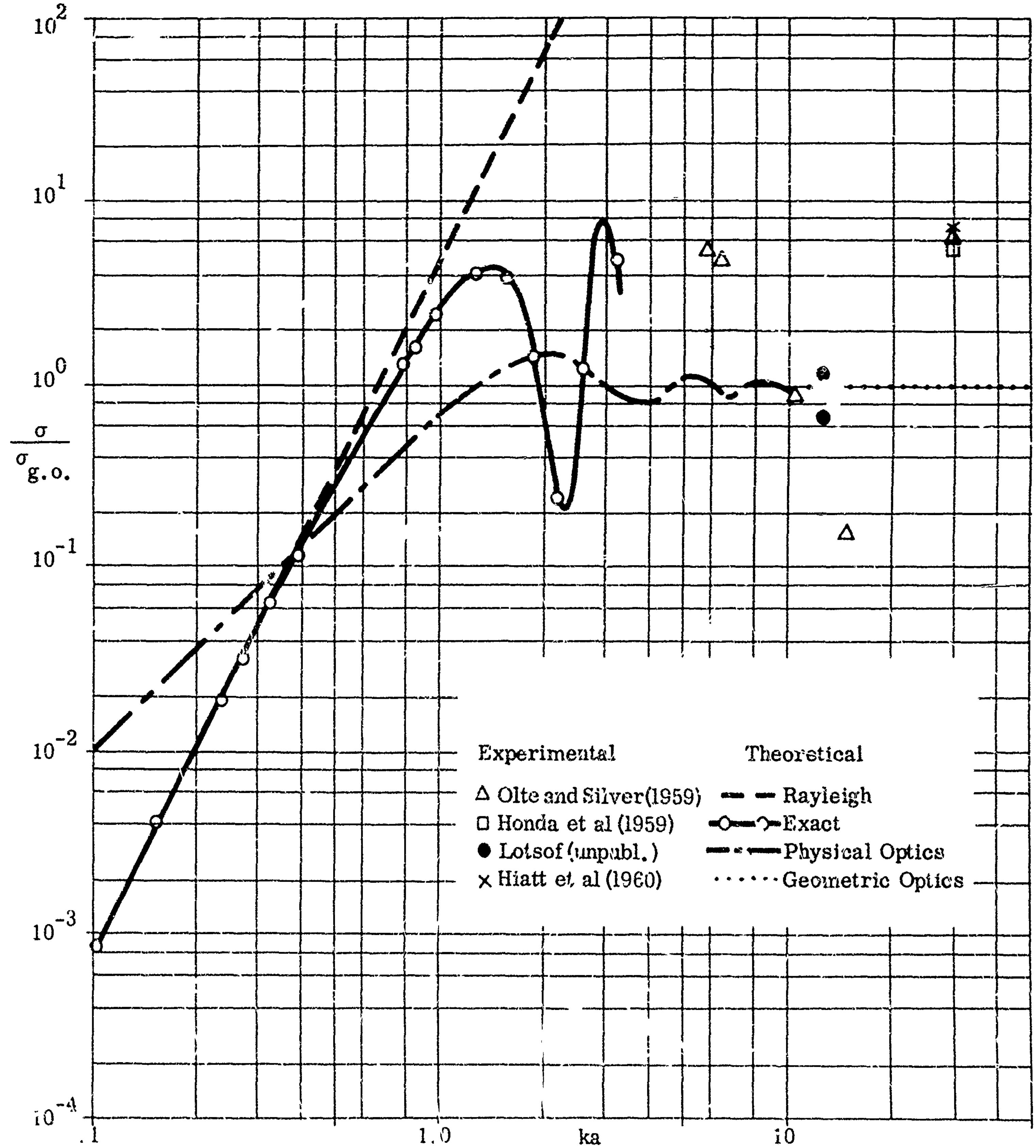


FIG. 24: NOSE-ON ELECTROMAGNETIC BACK SCATTERING CROSS SECTION OF A 10:1 PROLATE SPHEROID

Back scattering measurements of a 10:1 perfectly conducting prolate spheroid were carried out at The University of California, Berkeley, using the image plane technique (Honda, et al, 1959). The operating frequency was 9346 Mc and for ka =29.8, complete polar diagrams of the back scattered field were obtained in the plane perpendicular to the incident electric field. The results for back scattered cross section near broadside are plotted in Fig. 25 . Also included is the theoretical geometric optics cross section calculated from equation (3.49). At nose-on incidence the cross section was measured as about 4 times (6 db) larger than that predicted by geometric optics (see Fig.24). There was some doubt as to the reliability of the measurements for aspects near nose-on because the extremely small values of the scattered field admitted the possibility that the measured return was dominated by a spurious signal.

Subsequently an improved version of the same experimental setup was employed to measure the back scattering cross section of a set of five different conducting prolate spheroids, all having a ratio of major to minor axis of 10:1 (Olte and Silver, 1959). Their results for broadside (E perpendicular to axis of symmetry) and nose-on incidence are given in the following table. The nose-on values are plotted in Fig. 24 and substantiate the results of Honda et al.

σ IN db RELATIVE TO 6 IN. DIA. SPHERE

a	6.0	3.0	2.111	1.263	1.184
Nose-on	-26.0	-48.0	-43.3	-40.6	-40.9
Broadside	2.4	-0.5	-4.3	-12.7	-13.5

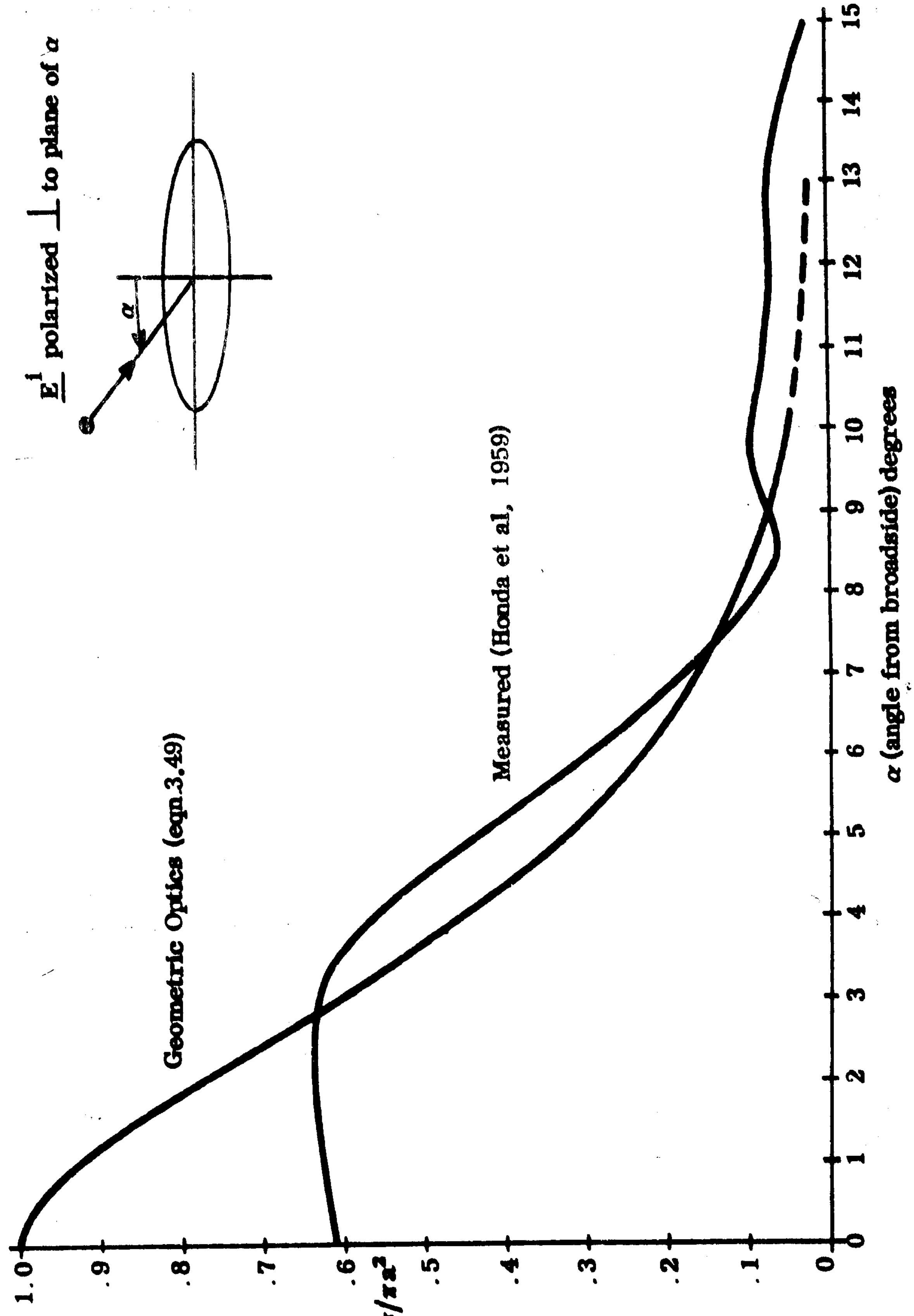


FIG. 25: BACK SCATTERING CROSS SECTION OF 10:1 PROLATE SPHEROID AS A FUNCTION OF ANGLE FROM BROADSIDE ASPECT. ka = 29.8

Further corroboration of the nose-on back scattering cross section of a 10:1 prolate spheroid for $ka = 29.8$ is given by measurements made at the Radiation Laboratory (Hiatt et al, 1960). The experiment was designed to measure the effect on back and forward scattering of coating various parts of a conducting spheroid with radar absorbing material. A perfectly conducting spheroid was also measured and the back scattering cross section appears in Fig. 24.

J. Lotsof of the Cornell Aeronautical Laboratory also measured the back scattering cross section of a 10:1 perfectly conducting prolate spheroid for various aspect angles. These data have not been published directly though they have appeared in the literature (Crispin et al, 1959; Siegel, 1959),[+] cited as a private communication. The data were measured at $ka = 12.56$ for both horizontal and vertical polarization. The results for horizontal polarization ($\underline{E}^i$ parallel to the plane of rotation) are given in Fig. 26 together with the theoretical result predicted by travelling wave theory (see Sec. 4.1.10). The results for vertical polarization ($\underline{E}^i$ perpendicular to the plane of rotation) are given in Fig. 27. The nose-on values in both cases have been renormalized and plotted in Fig. 24.

Some bistatic measurements were carried out on a 2:1 conducting prolate spheroid for incidence along the axis of symmetry by Rabinowitz (1956). Measurements were made at bistatic angles between 90^o and 180^o for both horizontal and vertical polarizations ($\underline{E}^i$ parallel and perpendicular to the plane of rotation) at a wavelength such that $ka = 103$. Quantitative results were not given but the qualitative scattered field behavior is evident in the results given in Fig. 28.

[+] Note that the ordinate scales in the graphs of these data in these references are too high by a factor of 10^4. Actually what is plotted is σ in cm^2, not m^2.

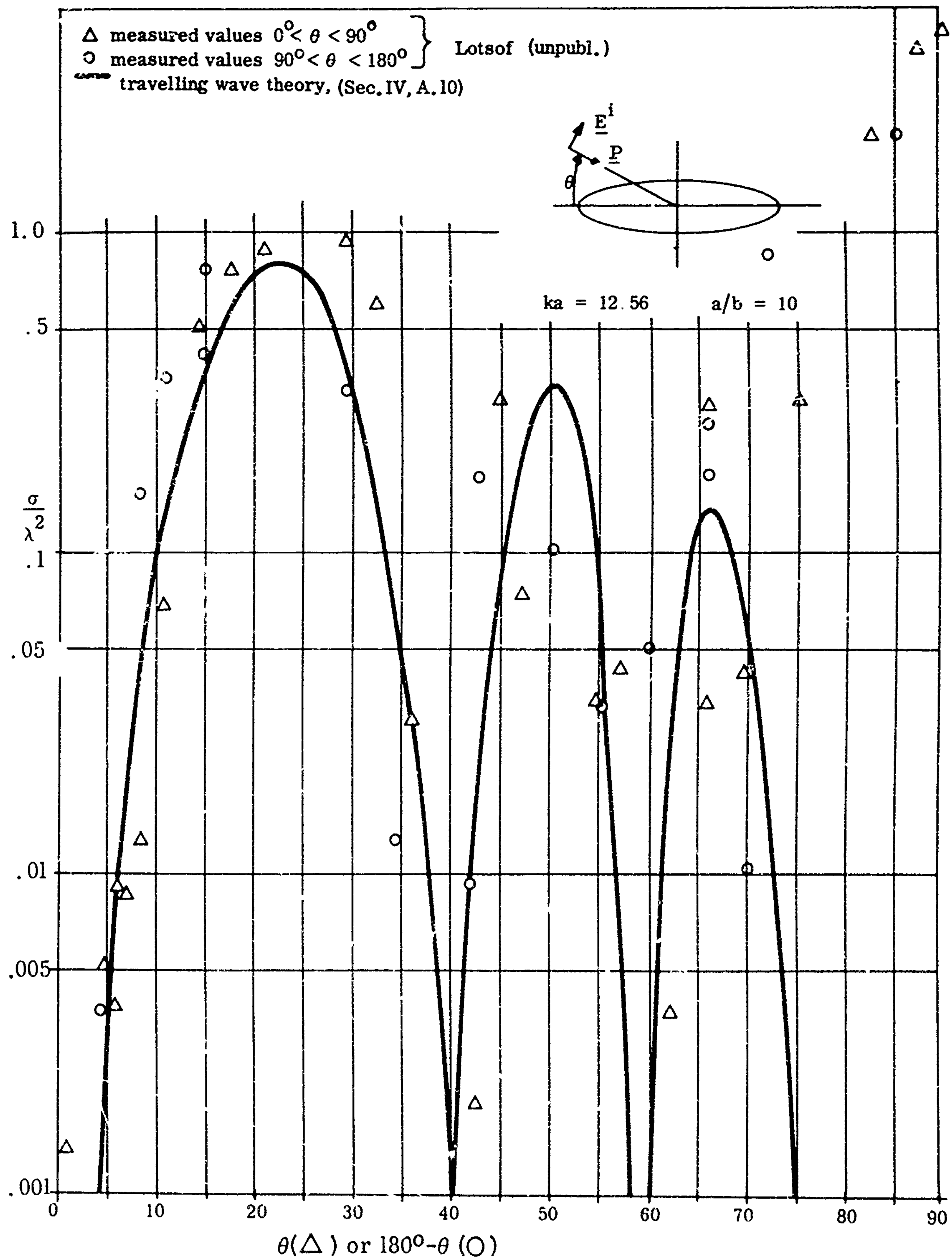


FIG. 26: BACK SCATTERING CROSS SECTION OF A 10:1 SPHEROID AS A FUNCTION OF ANGLE FROM SYMMETRY AXIS

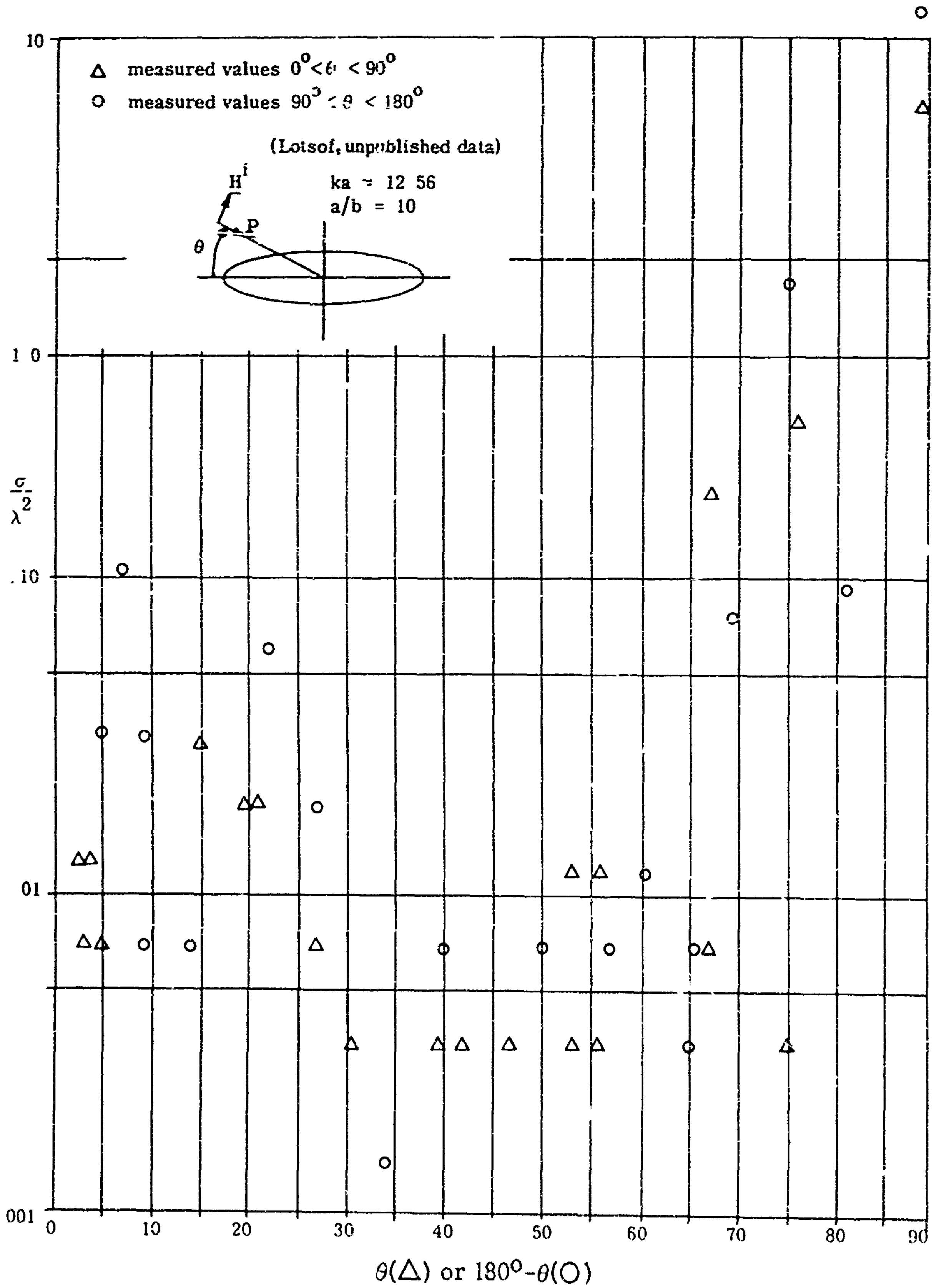


FIG 27: BACK SCATTERING CROSS SECTION OF A 10:1 SPHEROID AS A FUNCTION OF ANGLE FROM SYMMETRY AXIS

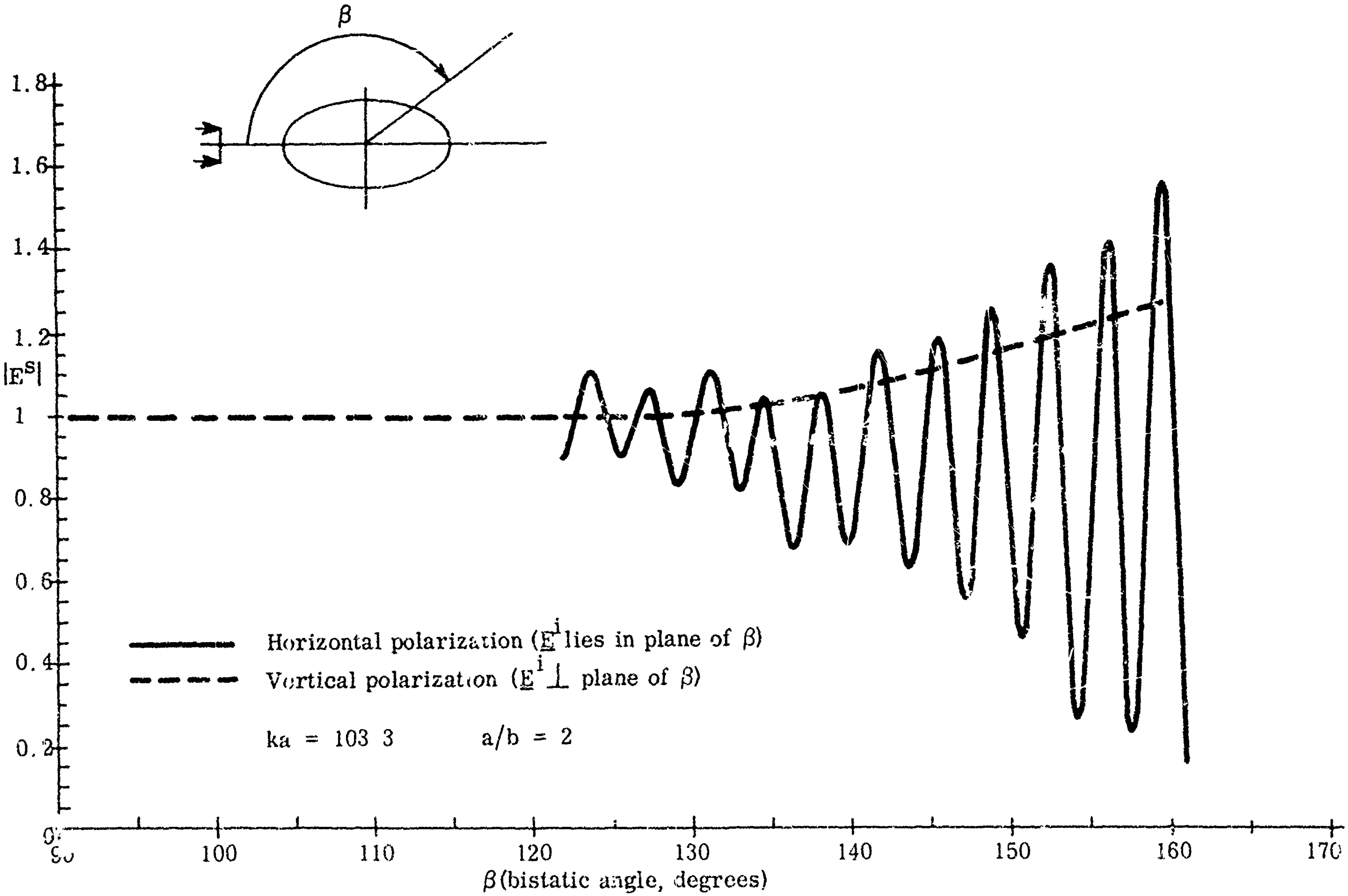


FIG 28: RELATIVE MAGNITUDE OF FIELD SCATTERED FROM A 2:1 PROLATE SPHEROID AS A FUNCTION OF RECEIVER ANGLE (Rabinowitz, 1956)

More extensive bistatic measurements were subsequently carried out at the Ohio State University experimental facility (Eberle and St. Clair, 1960). For bistatic angles of 0° (back scattering), 30°, 60°, 90°, 120°, and 140°, scattering cross section was measured continuously as a function of aspect for both horizontal and vertical polarization of transmitter and receiver. As indicated in Fig. 29, the experimental set-up involved fixing transmitter and receiver at a particular angular separation β, and rotating the target in a plane containing the transmitter

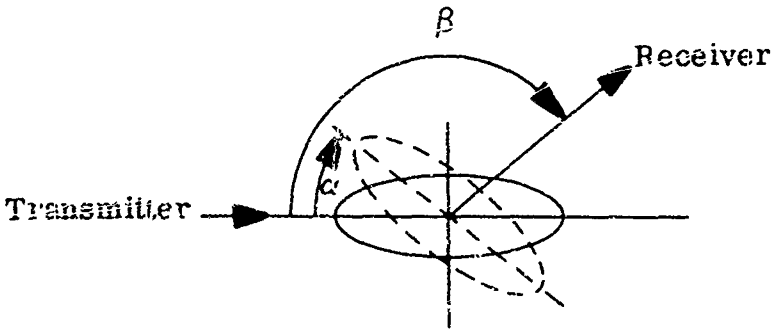


FIG. 29

and receiver directions and the spheroid axis of symmetry. The system operated at a wavelength of 3.2 cm and the spheroid (again perfectly conducting) had an axis ratio of 2.178 with $ka = 9.13$. In Fig. 30, the measured values of σ / λ^2 in db are plotted against aspect angle α. No attempt has been made to renormalize the data since as originally presented, the scale is too small to be read with much accuracy.

The experimental facility at Ohio State University was also used to measure the nose-on back scattering cross section of a dielectric spheroid (Thomas, 1962). For spheroids of axis ratio 1.35:1 and relative permittivity 1.8 (index of refraction 1.34)

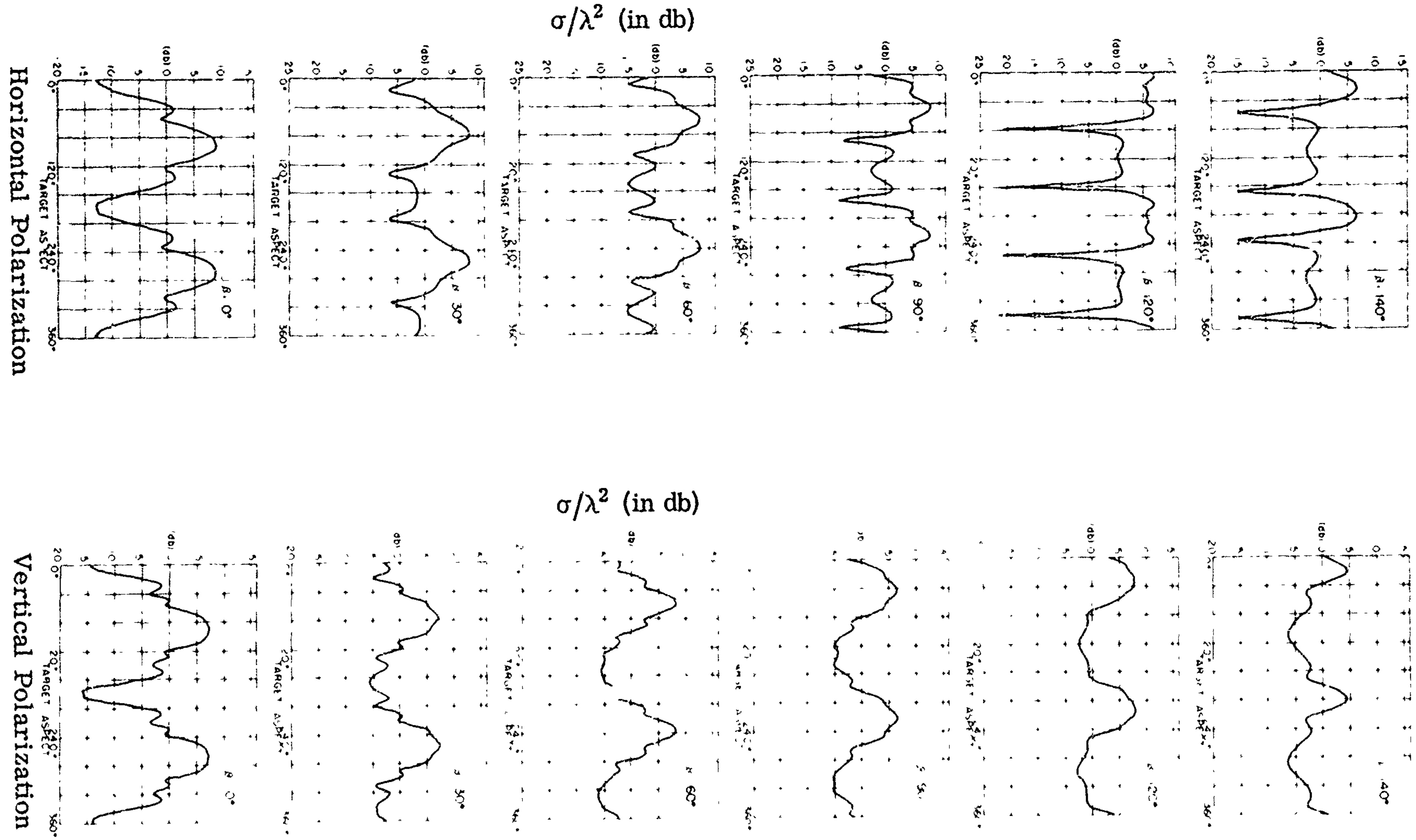


FIG. 30: BISTATIC CROSS SECTION OF PROLATE SPHEROID (Eberle and St. Clair, 1960). ka = 9.13, a/b=2.178 (Target aspect angle α and bistatic angle β depicted in Fig. 29).

the cross section was measured for various values of ka. A theoretical curve based on a modified geometric optics analysis was presented along with these results, but there appear to be discrepancies in the numerical work which have not been resolved at the date of this writing. The curve is therefore omitted here, but the experimental results are given approximately (as read from the published graph) in the following table, where b is the semi-minoraxis of the spheroid and the ratio $\sigma/\pi b^2$ is given in decibels.

b/λ =	.38	.40	.42	.76	.81	.85	.92	2.02
$\sigma/\pi b^2$ =	-6.7	-5.2	-2.7	-2.[illegible]	-2.3	-.04	-2.1	-0.7

More extensive measurements of scattering by dielectric spheroids have been carried out at Rensselaer Polytechnic Institute (Greenberg et al, 1961; 1963a, b). Measurements of scattering efficiency, $Q = \sigma_T/A$, A = geometric cross sectional area (see van de Hulst, 1957; Goodrich et al, 1961) were made on a spheroid of axis ratio 2:1 for a number of indices of refraction, $n = m - i\delta$, both real ($\delta = 0$) and complex ($\delta \neq 0$). Differential scattering cross sections were measured but not reported, and the total cross section was determined by measuring the forward scattered field. Measurements were made for incidence nose-on and broadside, the latter for both vertical and horizontal polarization. The results are given in Figs. 31-33.

A series of back scattering measurements on spheroids of small eccentricity was undertaken at the Ohio State University in support of the theoretical work of

Mushiake (1956). Only preliminary results were given, and these are shown in Fig. 34 . There was some question regarding the reliability of these results since the experiment was not readily reproducible. The refinement of the experiment was to be the subject of future work; however, at the present writing, refined results are still unavailable.

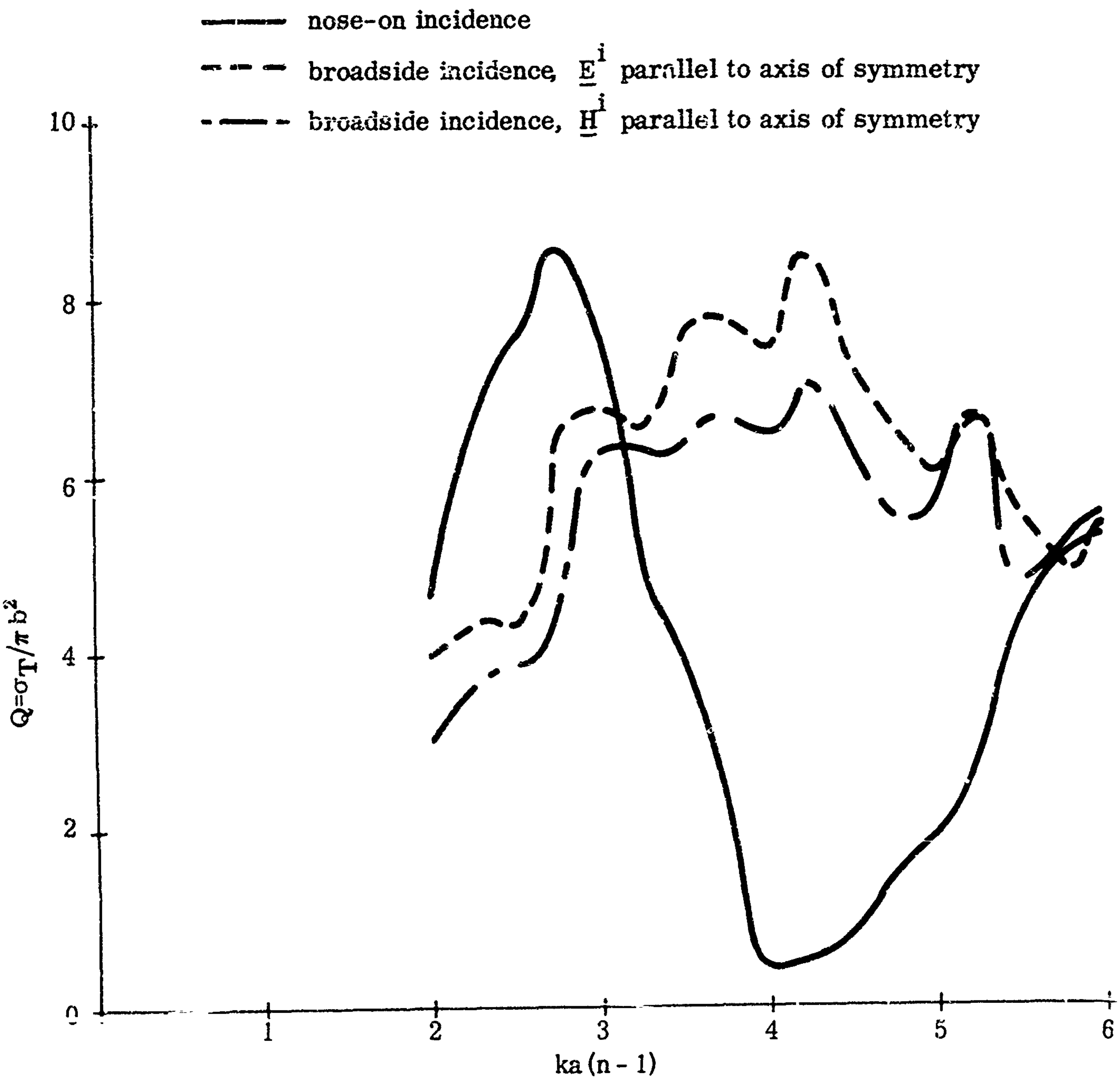


FIG 31: TOTAL CROSS SECTION OF A DIELECTRIC SPHEROID
n = 1 603, a/b = 2 (Greenberg et al, 1961)

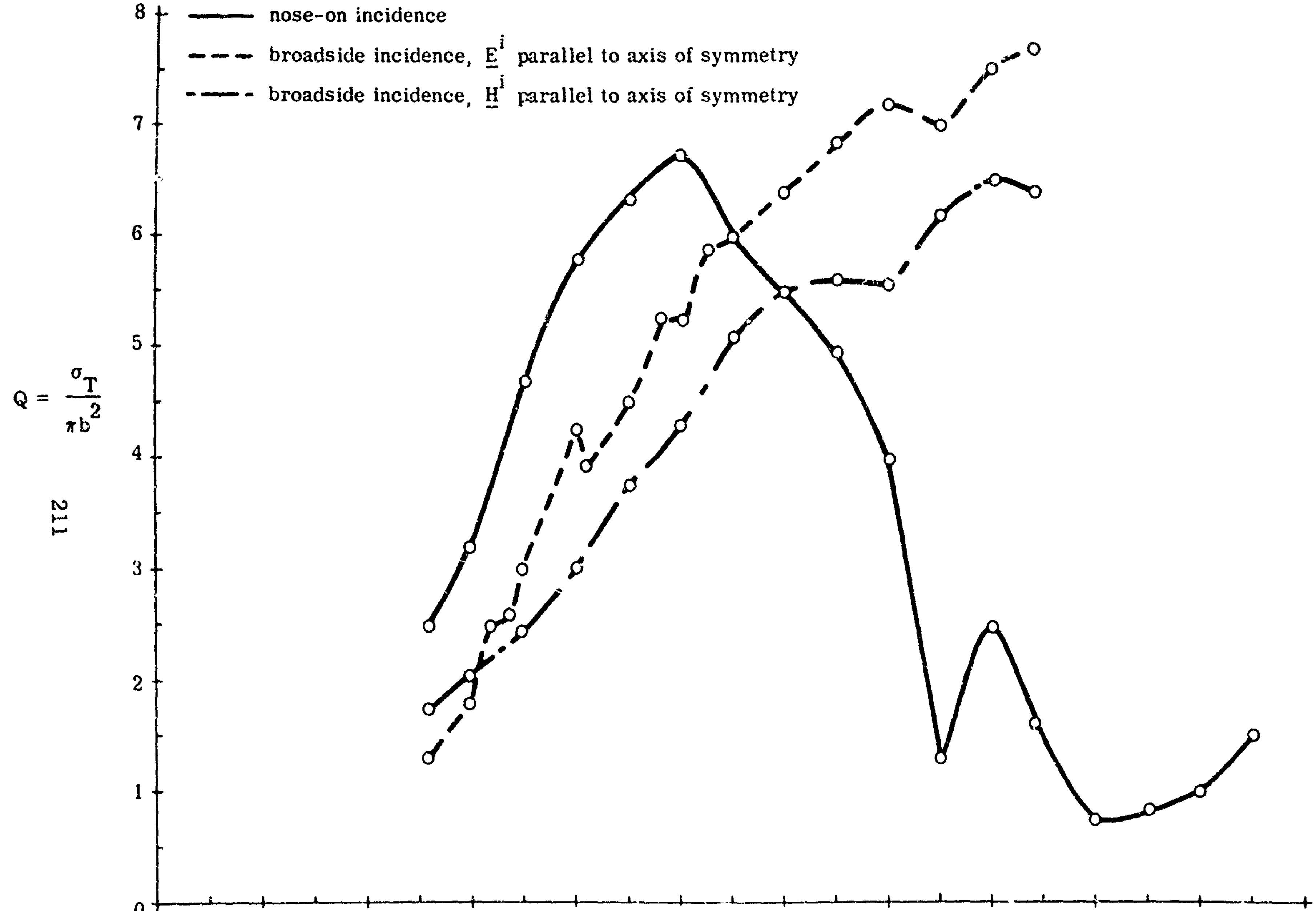


FIG 32: TOTAL CROSS SECTION OF A DIELECTRIC SPHEROID
n = 1.26, a/b = 2 (Greenberg et al. 1963b)

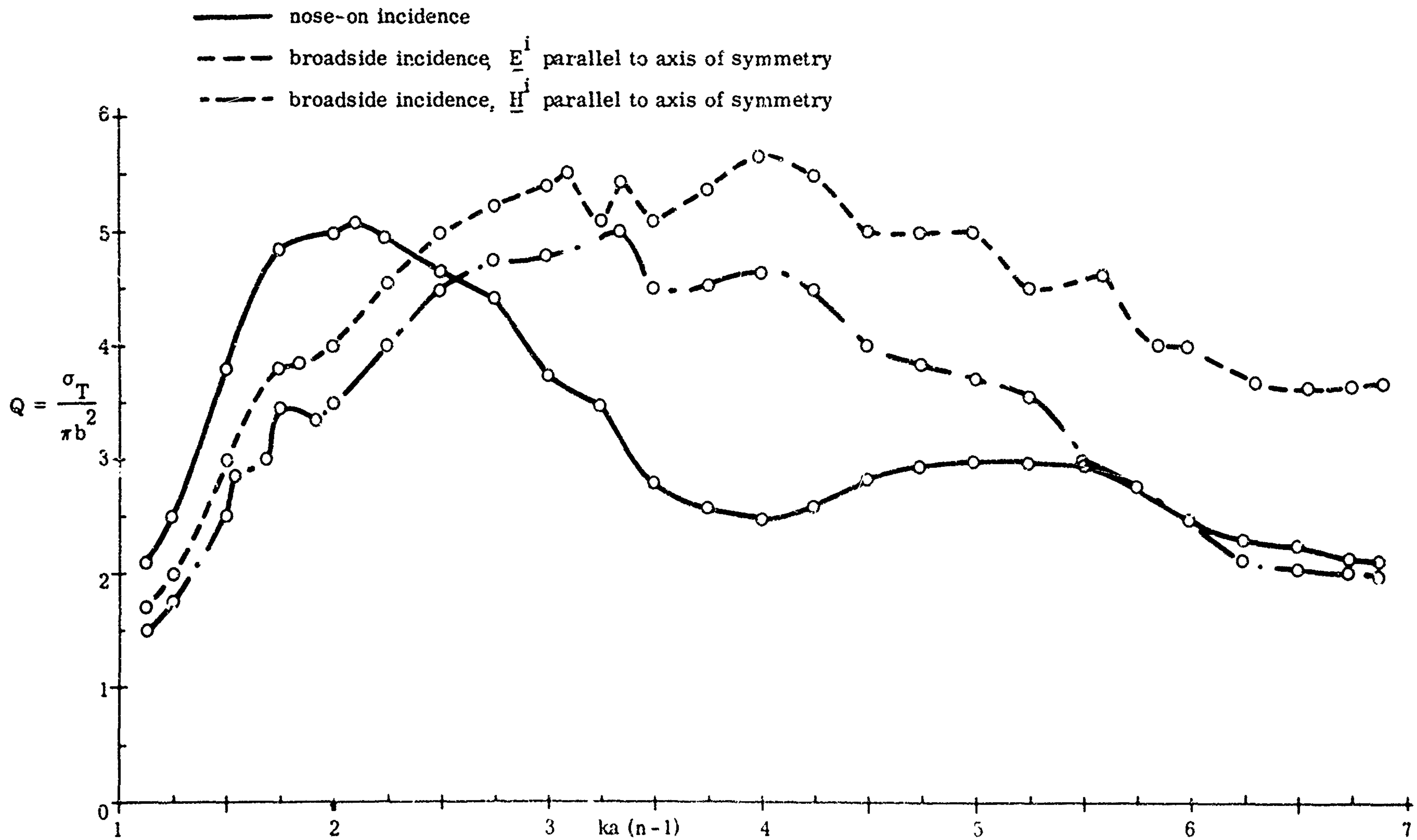


FIG 33: TOTAL CROSS SECTION OF A COMPLEX DIELECTRIC SPHEROID
$n = (m - i\delta) = 1.33 - .05i$ (Greenberg et al, 1963b)

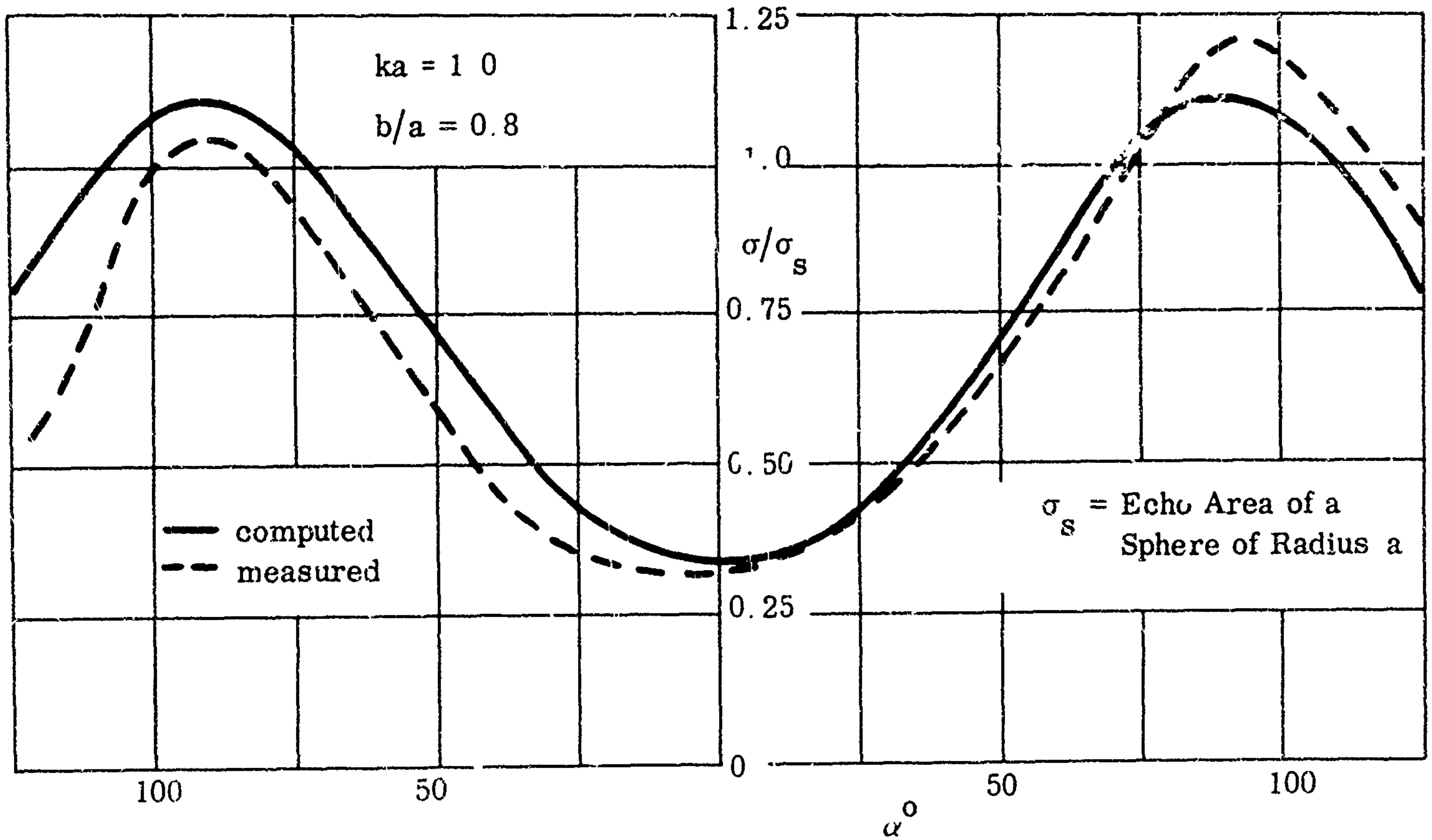


FIG 34: A COMPARISON BETWEEN COMPUTED AND MEASURED BACK SCATTERING CROSS SECTION OF A SPHEROID (α = 0 is nose-on) (Mushiake, 1956)

APPENDIX

INDEX TO NUMERICAL TABLES

The following is a guide to the principal numerical tables which have been computed in connection with the spheroid problem. Note that the terminology and normalizations used in the various sources are not uniform. To reconcile the different systems of notation, see the precise definitions in each source and the Table of Notations in Flammer (1957). The notation a(Δ)b in the third column below indicates that a quantity ranges from a to b inclusive at intervals of Δ. Accuracy is specified in significant figures.

Quantity	Source	Parameters, Arguments, Indices	Accuracy
Eigenvalue $\lambda_{mn}(c)$	Stratton et al (1956)	m = 0(1)8; n = m(1)8	7
	Flammer (1957)	a) m = 0(1)3; n = m(1)3; c = 0(.2)5.0	6-7
		b) m = 1; n = 5(2)19; c = 1.2, $\pi/2$, 2.0, $3\pi/4$, 2.5, 2.8, 3.0, π, 3.2	10
	Weeks (1959)	m = 1; n = 1(1)27 - 80(depending on c, see below) c = $\pi/2$ π $3\pi/4$ 2π 12 4π 5π 16 max.n = 28 39 48 44 45 80 62	9-10
	U. of M.Rad. Lab. unpublished	m = 0, 1; n = 0(1)3; c = .0935, .1043, .156, .234, .260, .312, .375, .521, .750, .780, .937, 1.251, 1.560, 1.876, 2.085, 2.493, 3.120, 3.75, 4.69, 5.86, 6.24, 10.43	15
Spheroidal Coefficients $d_k^{mn}(c)$	Stratton et al (1956)	m = 0(1)8; n = m(1)8; c = 0(.1)1.0(.2)8.0	7
	Flammer (1957)	a) m = 0(1)3; n = m(1)3; c = 0(.2)5.0 k = -2m(2) vanishing point	⩾ 5

Appendix

Index to Numerical Tables (cont.)

Quantity	Source	Parameters, Arguments, Indices	Accuracy
Spheroidal Coefficients $d_k^{mn}(c)$ (cont.)	Flammer (1957)	b) $m=1$; $n=5(2)13$; $c=1.2, \pi/2, 2.0, 3\pi/4, 2.5, 2.8, 3.0, \pi, 3.2$ $k = -2(2)$vanishing point	$\geqslant 8$
	U. of M. Rad. Lab. unpublished	a) $m=0,1$; $n=0(1)3$; $c=$all values specified above for eigenvalues; $k=$all necessary values between -16 and +16.	15
		b) $m=0$; $n=0(1)8$; $c\simeq .994, 1.391, 1.591, 1.999, 2.086, 2.185, 2.238, 2.782, 2.981, 2.998, 3.581, 3.681, 3.780, 3.80, 3.88, 4.00, 4.28, 4.60$	8
Angular Functions $S_{mn}(c,\eta)$	Flammer (1957)	$m=0(1)3$; $n=m(1)3$; $c=.5(.5)5.0$ $\theta \equiv \cos^{-1}\eta = 0^o(5^o)90^o$	$\geqslant 4$
	Spence (1951)	$m=0(1)3$; $n=m(1)(3-m)$ a) $c=1(1)5$; $\theta \equiv \cos^{-1}\eta=0^o(5^o)90^o$ b) $c=.5(.5)5.0$; $\theta=0^o, 30^o, 60^o, 90^o$	4
	Weeks (1959)	$m=1$; $n=1(1)20$; $c=\pi/2, 5, 8, 12$ $\theta \equiv \cos^{-1}\eta=5^o(5^o)90^o$	$\geqslant 9$
Radial Functions $R_{mn}^{(j)}(c,\xi)$ and derivatives with respect to ξ	Flammer (1957)	a) $j=1$; $m=0(1)3$; $n=m(1)3$; $c=.5(.5)5.0$ $\xi=1.005, 1.020, 1.044, 1.077$	4
		b) $j=1,2$; $m=1$; $n=1(2)13$ $c=1.2-3.2$ (9 values listed above) $\xi=1.01, 1.0001, 1.000001, 1.00000001$	$\geqslant 6$
		c) $j=2$; $m=0(1)3$; $n=m(1)3$; $c=1(1)5$ $\xi=1.005, 1.020, 1.044, 1.077$	4
	U. of M. Rad. Lab. unpublished	$j=1, 2$; $m=0,1$; $n=0(1)3$, $\xi=1.005$	15
	Mathur and Mueller (1955)	$j=1, 2, 4$; $m=0,1$; $n=0, 1, 2$ $c=.1, .2, .4, .6, .8$; $\xi=1.1, 1.2, 1.3$	5

Appendix

Index to Numerical Tables (cont.)

Quantity	Source	Parameters, Arguments, Indices	Accuracy
Radial Functions $R^{(j)}_{mn}(c, \xi)$ (cont.)	Weeks (1959)	$j = 1, 2$; $m = 1$; $n = 1(1)12$-22 (see below) $\xi = 1.077,\ 1.100$ $c = 5 \quad 8 \quad 12$ max. $n = 12 \quad 16 \quad 22$	3-9
Normalization constant $N_{mn}(c)$	Flammer (1957)	$m = 0(1)3$; $n = m(1)3$; $c = .5(.5)5.0$	4
	Mathur and Mueller (1955)	$m = 0$; $n = 0, 1, 2$; $c = .1, .2, .4, .6, .8$	5
Joining Factor $k^{(j)}_{mn}(c)$	Flammer (1957)	a) $j = 1$; $m = 0(1)3$; $n = 0(1)3$; $c = 1(1)5$	4
		b) $j = 1$; $m = 1$; $n = 1(2)13$; $c = 1.2$-3.2 (9 values listed above)	8
		c) $j = 2$; $m = 1$; $n = 1(2)7$; $c = 1.2$-3.2 (9 values listed above)	6

TABLE OF NOTATION

In a work of this sort, in which a large number of symbols are employed, some duplication is inevitable. Many of the quantities however are adventitious and are defined at the point of introduction and soon abandoned. Those in widespread or repeated use are listed below, along with brief definitions and references to their points of appearance in the text.

Symbol	Name of Quantity	Defined On Page
$A(q)$	Airy function	157
a	Semi-major axis of spheroid	7
b	Semi-minor axis of spheroid	7
A_{mn}, B_{mn}	Field expansion coefficients	27
c	1/2 x Wave number x interfocal distance = kF	13
$d_r^{mn}(c)$	Spheroidal coefficient	18
$\underline{E}^{i,s}$	Incident, scattered electric fields	48
E_ξ, E_η, E_ϕ	Components of electric field in spheroidal coordinates	49
F	1/2 x interfocal distance	7
$G_0(\underline{r}, \underline{r}')$	Free-space Green's function	27
$G(\underline{r}, \underline{r}')$	Green's function of particular body with point source	29

TABLE OF NOTATION(CONT.)

Symbol	Name of Quantity	Defined On Page
$G_{\infty}(\underline{r}, \underline{r}')$	Fundamental solution with plane wave excitation	30
$\underline{H}^{i,s}$	Incident, scattered magnetic fields	48
$H_{\xi}, H_{\eta}, H_{\phi}$	Components of magnetic field in spheroidal coordinates	49
$h_{\xi}, h_{\eta}, h_{\phi}$	Metric coefficients of spheroidal coordinates	10
$h_n^{(1,2)}$	Spherical Hankel function of 1st, 2nd kind	22
$K(\xi, \eta)$	Kernel function in integral representation	21
$k_{mn}^{(1)}(c)$	Proportionality factor of radial and angular functions	22
$\underline{L}_{\substack{e\\o}mn}, \underline{M}_{\substack{e\\o}mn}, \underline{N}_{\substack{e\\o}mn}$	Hansen's vector wave functions	47
ℓ_j, m_j, n_j	Direction cosines of vector identified by index	146
N_{mn}	Normalization constant for angular functions	20
n	Index of refraction	208
P_n^m	Associated Legendre function of order m, degree n, first kind	18
$\underline{P}$	Poynting vector	189
p	Dipole strength	38
Q_n^m	Associated Legendre function, 2nd kind	18

TABLE OF NOTATION (CONT.)

Symbol	Name of Quantity	Defined On Page
$q_n^{(1,2)}$	Zero of Airy function or its derivative	157
$R_{mn}^{(j)}(c,\xi)$	Radial spheroidal function	21
R	Distance between two points in space	38
$\underline{r}$, r	Radius vector, magnitude of same	27
$S_{mn}(c,\eta)$	Angular spheroidal function, first kind	18
s	Distance along ray	103
$T_n^1(\cos\theta)$	Gegenbauer function of order 1	174
t	Time variable	27
$u(P)$	Scalar field strength at point P	103
V	Volume of spheroid	68
x, y, z	Cartesian coordinates	9
$z_n^{(j)}(kr)$	General spherical Bessel function	22
α	Incident Angle	120
β	Separation angle between transmitter and receiver	90
$\Delta(1, 2)$	Wronskian determinant	22
∇^2	Laplacian operator	13

TABLE OF NOTATION (CONT.)

Symbol	Name of Quantity	Defined On Page
∇	Gradient operator	11
δ_{mn}	Kronecker delta	139
ϵ	Permittivity	33
ϵ	Eccentricity of spheroid $= \frac{1}{\xi}$	161
ϵ_m	Neumann number	28
η	Angular spheroidal coordinate	7
Θ	Angle between vector $\underline{R}$ and dipole axis	38
θ	Spherical or spheroidal polar angle	7
λ	Wavelength	13
$\lambda_{mn}(c)$	Eigenvalue of spheroidal equation	13
μ	Permeability of medium	33
$\mu_{m,n}$	Parity modulus	139
ν	Perturbation quantity	121
ξ	Radial spheroidal coordinate	7
ξ_o	Coordinate of scattering surface	29
π_ϵ, π_m	Electric, magnetic Hertz potentials	67
ρ_{mn}	Normalization factor for radial function	22

TABLE OF NOTATION(CONT.)

Symbol	Name of Quantity	Defined On Page
σ	Scattering cross section	70
$\sigma_{g.o.}$	Geometrical optics scattering cross section	90
ϕ	Azimuthal variable	7
ϕ_{emn}	Spheroidal harmonic	76
ψ	Wave function	27
ω	Angular frequency	27

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